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MODULE 1

INTRODUCTION TO MATHEMATICAL ECONOMICS

Mathematical Economics: Meaning and Importance, Mathematical Representation of Economic Models, Economic functions: Demand function, Supply function, Utility function, Consumption function, Production function, Cost function, Revenue function, Profit function, Saving function, Investment function

1.1 MATHEMATICAL ECONOMICS

Mathematical economics is a branch of economics that engages mathematical tools and methods to analyse economic theories. Mathematical economics is best defined as a sub-field of economics that examines the mathematical aspects of economies and economic theories. Or put into other words, mathematics such as calculus, matrix algebra, and differential equations are applied to illustrate economic theories and analyse economic hypotheses.

It may be interesting to begin the study of mathematical economics with an enquiry into the history of mathematical economics. It is generally believed that the use of mathematics as a tool of economics dates from the pioneering work of Cournot (1838). However there were many others who used mathematics in the analysis of economic ideas before Cournot. We shall make a quick survey of the most important contributors.

Sir William Petty is often regarded as the first economic statistician. In his Discourses on Political Arithmetic (1690), he declared that he wanted to reduce political and economic matters to terms of number, weight, and measure. The first person to apply mathematics to economics with any success was an Italian, Giovanni Ceva who in 1711 wrote a tract in which mathematical formulas were generously used. He is generally regarded as the first known writer to apply mathematical method to economic problems. The Swiss mathematician Daniel Bernoulli in 1738 for the first time used calculus in his analysis of a probability that would result from games of chance rather than from economic problems.

Among the early French writers who made some use of mathematics was Francois de Forbonnais who used mathematical symbols, especially for explaining the rate of exchange between two countries and how an equilibrium is finally established between them. He is best known for his severe attack on Physiocracy.

While this is a long list to those who showed curiosity in the use of mathematics in economics, interestingly J. B. Say showed little or no interest in the use of mathematics. He did not favour the use of mathematics for explaining economic principles. A German who is reasonably well known, especially in location theory, is Johann Heinrich von Thunen. His first work, The Isolated State (1826), was an attempt to explain how transportation costs influence the location of agriculture and even the methods of cultivation.

The French engineer, A. J. E. Dupuit, used mathematical symbols to express his concepts of supply and demand. Even though he had no systematic theory he did develop the concepts of utility and diminishing utility, which were clearly stated and presented in graphical form.

He viewed the price of a good as dependent on the price of other goods. Another Lausanne economist, Vilfredo Pareto, ranks with the best in mathematical virtuosity. The Pareto optimum and the indifference curve analysis (along with Edgeworth in England) were clearly conceived in the mathematical frame work.

A much more profound understanding of the analytical power of mathematics was shown by Leon Walras. He is generally regarded as the founder of the mathematical school of economics. He set out to translate pure theory into pure mathematics. After the pioneering work of Jevons (1871) and Walras (1874), the use of mathematics in economics progressed at very slow pace for a number of years. The first of the latter group is F. Y. Edgeworth. His contribution to mathematical economics is found mainly in the 1881 publication in which he dealt with the theory of probability and statistical theory.

F. Y. Edgeworth’s contribution to mathematical economics is found mainly in the 1881 publication in which he dealt with the theory of probability, statistical theory and the law of error. The indifference analysis was first propounded by Edgeworth in 1881 and restated in 1906 by Pareto and in 1915 by the Russian economist Slutsky, who used elaborate mathematical treatment of the topic. Alfred Marshall made extensive use of mathematics in his Principles of Economics (1890). His interest in mathematics dates from his early schooldays when his first love was for mathematics, not the classics.” In the Principles he used mathematical techniques very effectively.

Another economist whose influence spans many years is the American, Irving Fisher. Fisher belongs to other groups as well as to the mathematical moderns. His life's work reveals him as a statistician, econometrician, mathematician, pure theorist, teacher, social crusader, inventor, businessman, and scientist. His contribution to statistical method, The Making of Index Numbers, was great. The well-known Fisher formula, M V + M' V' = P T, is an evidence of his contribution to quantitative economics.

A strong inducement to formulate economic models in mathematical terms has been the post-World War II development of the electronic computer. In the broad area of economics there has been a remarkable use of mathematical techniques. Economics, like several other disciplines, has always used quantification to some degree. Common terms such as wealth, income, margins, factor returns, diminishing returns, trade balances, balance of payments, and the many other familiar concepts have a quantitative connotation. All economic data have in some fashion been reduced to numbers which became generally known as economic statistics.

The most noteworthy developments, however, have come in the decades since about 1930. It was approximately this time that marked the ebb of neoclassicism, the rise of institutionalism, and the introduction of aggregate economics. Over a period of years scholars have developed new techniques designed to help in the explanation of economic behavior under different market situations using mathematics. Now we discuss some of the economic theories and the techniques of modern analysis. They are given largely on a chronological basis, and their significance is developed in the discussion.

The Theory of Games: The pioneering work was done by John von Neumann in 1928. The theory became popular with the publication of Theory of Games and Economic Behaviour in 1944. Basically, The Game Theory holds that the actions of players in gambling games are similar to situations that prevail in economic, political, and social life. The theory of games has many elements in common with real-life situations. Decisions must be made on the basis of available facts, and chances must be taken to win. Strategic moves must be concealed (or anticipated) by the contestants based on past knowledge and future estimates. Success or failure rests, in large measure, on the accuracy of the analysis of the elements. The theory of games introduced an interesting and challenging concept. Economists have made some use of it, and it has also been fitted into other social sciences, notably sociology and political science.

Linear Programming: Linear programming is a specific class of mathematical problems in which a linear function is maximized (or minimized) subject to given linear constraints.The founders of the subject are generally regarded as George B. Dantzig, who devised the simplex method in 1947, and John von Neumann, who established the theory of duality that same year. The scope of linear programming is very broad. It brings together both theoretical and practical problems in which some quantity is to be maximized or minimized. The data could be almost any fact such as profit, costs, output, distance to or from given points, time, and so on. It also makes allowance for given technology and restraints that may occur in factor markets or in finance. Linear programming has been proven very useful in many areas. It is in common use in agriculture, where chemical combinations of proper foods for plants and animals have been worked out, and in the manufacture of many processed agricultural products. It is necessary in modem materials scheduling, in shipping, and in final production. The Nobel prize in economics was awarded in 1975 to the mathematician Leonid Kantorovich (USSR) and the economist Tjalling Koopmans (USA) for their contributions to the theory of optimal allocation of resources, in which linear programming played a key role.

Input-Output Analysis: In terms of techniques, input-output analysis is a rather special case of linear programming. It was devised originally by Leontief and, in a sense, was a World War II-inspired analysis. Basically it was designed for presenting a general equilibrium theory suited for empirical study. The problem is to determine the interrelationship of sector inputs and outputs on other sectors or on all sectors which use the product.The rational for the term IOA can be explained like this. There is a close interdependence between different sectors of a modern economy. This interdependence arises out of the fact that the output of any given industry is utilized as an input by the other industries and often by the same industry itself. Thus the IOA analyses the interdependence between different sectors of an economy. The basis of IOA is the input - output table which can be expressed in the form of matrices.

1.1.1 Mathematical Economics: Meaning and Importance

Mathematical economics is the application of mathematical methods to represent economic theories and analyse problems posed in economics. It allows formulation and derivation of key relationships in a theory with clarity, generality, rigor, and simplicity. By convention, the methods refer to those beyond simple geometry, such as differential and integral calculus, difference and differential equations, matrix algebra, and mathematical programming and other computational methods.

Mathematics allows economists to form meaningful, testable propositions about many wide-ranging and complex subjects which could not be adequately expressed informally.

Further, the language of mathematics allows economists to make clear, specific, positive claims about controversial or contentious subjects that would be impossible without mathematics. Much of economic theory is currently presented in terms of mathematical economic models, a set of stylized and simplified mathematical relationships that clarify assumptions and implications.

Paul Samuelson argued that mathematics is a language. In economics, the language of mathematics is sometimes necessary for representing substantive problems. Moreover, mathematical economics has led to conceptual advances in economics.

1.1.2 Advantages of Mathematical economics

(1) The 'language' used is more concise and precise (2) a number of mathematical theorems help us to prove or disprove economic concepts (3) helps us in giving focus to the assumptions used in economics (4) it make the analysis more rigours (5) it allows us to treat the general n-variable case, otherwise the number of variables in economic analysis will be very limited.

However, you should also understand that there are economists who criticise that a mathematically derived theory is unrealistic. This certainly means the untimely use of mathematics may be worthless. Perhaps Alfred Marshall is the best person to quote on the cautions on using mathematics in economics. Though Marshall made extensive use of mathematics in his Principles of Economics, he was not convinced that economics so written would be read or understood. In 1898 he wrote, "The most helpful applications of mathematics to economics are those which are short and simple, which employ few symbols; and which aim at throwing a bright light on some small part of the great economic movement rather than at representing its endless complexity." He held to a rule "to use mathematics as a shorthand language rather than an engine of inquiry."

1.1.3 Mathematical Representation of Economic Models

Economic models generally consist of a set of mathematical equations that describe a theory of economic behaviour. The aim of model builders is to include enough equations to provide useful clues about how rational agents behave or how an economy works. An economic model is a simplified description of reality, designed to yield hypotheses about economic behaviour that can be tested. An important feature of an economic model is that it is necessarily subjective in design because there are no objective measures of economic outcomes. Different economists will make different judgments about what is needed to explain their interpretations of reality.

There are two broad classes of economic models - theoretical and empirical. Theoretical models seek to derive verifiable implications about economic behaviour under the assumption that agents maximize specific objectives subject to constraints that are well defined in the model. They provide qualitative answers to specific questions - such as the implications of asymmetric information (when person on one side of a transaction knows more than the other person) or how best to handle market failures.

In contrast, empirical models aim to verify the qualitative predictions of theoretical models and convert these predictions to precise, numerical outcomes. The validity of a model may be judged on several criteria. Its predictive power, the consistency and realism of its assumptions, the extent of information it provides, its generality (that is, the range of cases to which it applies) and its simplicity.

1.2 ECONOMIC FUNCTION

A variable represents a concept or an item whose magnitude can be represented by a number, i.e. measured quantitatively. Variables are called variables because they vary, i.e. they can have a variety of values. Thus a variable can be considered as a quantity which assumes a variety of values in a particular problem. Many items in economics can take on different values. Mathematics usually uses letters from the end of the alphabet to represent variables. Economics however often uses the first letter of the item which varies to represent variables. Thus p is used for the variable price and q is used for the variable quantity.

An expression such as $4x^3$ is a variable. It can assume different values because x can assume different values. In this expression x is the variable and 4 is the coefficient of x. Coefficient means 4 works together with x. Expressions such as $4x^3$ which consists of a coefficient times a variable raised to a power are called monomials. A monomial is an algebraic expression that is either a numeral, a variable, or the product of numerals and variables. (Monomial comes from the Greek word, *monos*, which means one.) Real numbers such as 5 which are not multiplied by a variable are also called monomials. Monomials may also have more than one variable. $4x^3y^2$ is such an example. In this expression both x and y are variables and 4 is their coefficient.

The following are examples of monomials: x, $4x^2$, $-6xy^2z$, 7

One or more monomials can be combined by addition or subtraction to form what are called polynomials. (Polynomial comes from the Greek word, poly, which means many.) A polynomial has two or more terms i.e. two or more monomials. If there are only two terms in the polynomial, the polynomial is called a binomial.

The expression $4x^3y^2 - 2xy^2 + 3$ is a polynomial with three terms.

These terms are $4x^3y^2$, $-2xy^2$, and 3. The coefficients of the terms are 4, −2, and 3.

The degree of a term or monomial is the sum of the exponents of the variables. The degree of a polynomial is the degree of the term of highest degree. In the above example the degrees of the terms are 5, 3, and 0. The degree of the polynomial is 5.

Remember that variables are items which can assume different values. A function tries to explain one variable in terms of another.

Independent variables are those which do not depend on other variables. Dependent variables are those which are changed by the independent variables. The change is caused by the independent variable. The independent variable is often designated by x. The dependent variable is often designated by y.

We say y is a function of x. This means y depends on or is determined by x.

Mathematically we write $y = f(x)$

It means that mathematically y depends on x. If we know the value of x, then we can find the value of y.

A function is a mathematical relationship in which the values of a single dependent variable are determined by the values of one or more independent variables. Function means the dependent variable is determined by the independent variable(s). A function tries to define these relationships. It tries to give the relationship a mathematical form. An equation is a mathematical way of looking at the relationship between concepts or items. These concepts or items are represented by what are called variables. Economists are interested in examining types of relationships. For example an economist may look at the amount of money a person earns and the amount that person chooses to spend. This is a consumption relationship or function. As another example an economist may look at the amount of money a business firm has and the amount it chooses to spend on new equipment. This is an investment relationship or investment function. Functions with a single independent variable are called univariate functions. There is a one to one correspondence. Functions with more than one independent variable are called multivariate functions.

Example of use of functions in solving problems:

y = f(x) = 3x + 4

This is a function that says that, y, a dependent variable, depends on x, an independent variable. The independent variable, x, can have different values. When x changes y also changes.

Find f(0). This means find the value of y when x equals 0.

f(0) = 3 times 0 plus 4

f(0) = 3(0) + 4 = 4

Find f(1). This means find the value of y when x equals 1.

f(1) = 3 times 1 plus 4

f(1) = 3(1) + 4 = 7

Find f(-1). This means find the value of y when x equals -1.

f(-1) = 3 times (-1) plus 4

f(1) = 3(-1) + 4 = 1

1.2.1 Demand function

Demand function express the relationship between the price of the commodity (independent variable) and quantity of the commodity demanded (dependent variable).It indicate how much quantity of a commodity will be purchased at its different prices. Hence, d_x represent the quantity demanded of a commodity and p_x is the price of that commodity. Then,

Demand function $\boldsymbol{d_x} = \mathbf{f}\ (\boldsymbol{p_x})$

The basic determinants of demand function

$\boldsymbol{Q_x} = \mathbf{f}\ (\boldsymbol{P_x}, \boldsymbol{P_r}, \mathbf{Y, T, W, E})$

Where Qx: quantity demanded of a commodity X, Px: price of commodity X, Pr: price of related good, Y: consumer’s income, T: Consumer/s tastes and preferences, W: Consumer’s wealth, E: Consumer’s expectations.

Example: $Q_d = p^2 - 20p + 125$

This is a function that describes the demand for an item where p is the dollar price per item. It says that demand depends on price.

Find the demand when one item costs Rs. 2

$d(2) = 2^2 - 20(2) + 125 = 89$

Find the demand when one item costs Rs. 5

$d(5) = 5^2 - 20(5) + 125 = 50$

Notice that the demand decreases as the price increases which you know is the law of demand.

1.2.2 Supply function

The functional relationship between the quantity of commodities supplied and various determinants is known as supply function. It is the mathematical expression of the relationship between supply and factors that affect the ability and willingness of the producer to offer the product.

Mathematically, a supply function can be expressed as

Supply $S_x = f(P_x)$

There are a number of factors and circumstances which can influence a producer's willingness to supply the commodity in the market. These factors are price of the commodity, price of the related goods, price of the factors of production, goal of producers, state of technology, miscellaneous factors (we can include factors such as means of transportation and communication, natural factors, taxation policy, expectations, agreement among the producers, etc.) Incorporating all such factors we may write the basic form of a supply function as;

$Q_s = f(G_f, P, I, T, P_r, E, G_p)$

Where Qs: quantity supplied, Gf: Goal of the firm, P: Product's own price, I: Prices of inputs, T: Technology, Pr: Prices of related goods, E: Expectation of producer's, Gp: government policy.

Example: Given a supply function $Q_s = -20 + 3P$ and demand function $Q_d = 220 - 5P$, Find the equilibrium price and quantity.

Equilibrium in any market means equality of demand and supply.

Hence, at equilibrium $-20 + 3P = 220 - 5P$.

Solving the above we get, Equilibrium price P = 30, Equilibrium quantity $Q_s = Q_d = 70$

1.2.3 Utility function

Utility function is a mathematical function which ranks alternatives according to their utility to an individual. The utility function measures welfare or satisfaction of a consumer as a function of consumption of real goods, such as food, clothing and composite goods rather than nominal goods measured in nominal terms. Thus the utility function shows the relation between utility derived from the quantity of different commodity consumed. A utility function for a consumer consuming three different goods may be represented:

$$U = f(X_1, X_2, X_3\ldots\ldots\ldots)$$

Example: Given the utility unction of a consumer $U = 2x^2 + 5$, find the marginal utility.

Marginal utility is given by the first order derivative of the total utility function.

$$\frac{dU}{dx} = 4x$$

2. Given a utility function for an individual consuming two goods x and y,

u = xy + 3x + 4y, find marginal utility of good x and good y.

$$MU_x = \frac{\partial u}{\partial x} = y + 3,\ MU_y = \frac{\partial u}{\partial y} = x + 4$$

1.2.4 Consumption Function

The consumption function refers to the relationship between income and consumption. It is a functional relationship between consumption and income. Symbolically, the relationship is represented as C= f(Y), where C is consumption, Y is income. Thus the consumption function indicates a functional relationship between C and Y, where C is the dependant variable and Y is the independent variable, i.e., C is determined by Y. In fact, propensity to consume or consumption function is a sketch of the various amounts of consumption expenditure corresponding to different levels of income.

In the Keynesian framework, the consumption function or propensity to consume, refers to a functional relationship between two aggregates, i.e., total consumption and gross national income. The **Keynesian Consumption function $C = a + bY_d$,** expresses the level of consumer spending depending on three factors as explained below.

Y_d = disposable income (income after government intervention – e.g. benefits, and taxes)

a = autonomous consumption (This is the level of consumption which does not depend on income. The argument is that even with zero income we still need to buy some food to eat, through borrowing or using our savings.)

b = marginal propensity to consume (also known as induced consumption).

The average propensity to consume is the ratio of consumption expenditure to any particular level of income. It is found by dividing consumption expenditure by income, or APC = C/Y. It is expressed as the percentage or proportion of income consumed. The marginal propensity to consume is the ratio of the change in consumption to the change in income. It can be found by dividing change in consumption by a change in income, or by finding the first derivative of the utility function. The MPC is constant at all levels of income.

The MPC is the rate of change in the APC. When income increases, the MPC falls but more than the APC. Contrariwise, when income falls, the MPC rises and the APC also rises but at a slower rate than the former. Such changes are only possible during cyclical fluctuations whereas in the short-run there is change in the MPC and MPC<APC.

Example: Given a consumption function C = 150 + 0.65Yd, autonomous consumption is _____ and MPC is_____. (*Ans. 150, 0.65)*

1.2.5 Saving function

The relationship between disposable income and saving is called the savings function. The saving function can represented in a general form as $S = f(Y)$, where S is saving, and Y is income, f is the notation for a generic, unspecified functional form. Because saving is the difference between disposable income and consumption, the saving function is a complementary relation to the consumption function. It is assumed that whatever is not consumed is saved. So MPC + MPS = 1.

Given a saving function S = 70 + 0.8Y, find MPS and MPC

MPS = 0.8, MPC = 1 – MPS = 0.2.

1.2.6 Production function

In a crude sense, production is the transformation of inputs into output. In another way, production is the creation of utility. Production is possible only if inputs are available and used. There are different types of inputs and economists classify them into land, labour, capital and organisation. In modern era, the meaning of these terms is redefined. Today, land covers all natural resources, labour covers human resources, capital is replaced by the term technology and finally, instead of organisation, we use the term management.

We explain production as the transformation of inputs into output.

In a general form a production function can be written as $Q = f(x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots x_n)$

Where Q represents the quantity produced, $x_1 \ldots\ldots x_n$ are inputs.

The general mathematical form of Production function is:

$$Q = f(L,K,R,S,v,e)$$

Where Q stands for the quantity of output, L is the labour, K is capital, R is raw material, S is the Land, v is the return to scale and e is efficiency parameters.

Example: : Q = 42KL – 3K2 – 2L2, Q = K0.4L0.5

1.2.7 Cost Function

Cost function expresses the relationship between cost and its determinants such as the level of output (Q), plant size (S), input prices (P), technology (T), managerial efficiency (E) etc. Mathematically it can be expressed as

$C = f(Q, S, P, T, E)$, where C is the unit cost or total cost

Example: Given the total cost function TC = $Q^2 + 8Q + 90$, find average cost (AC) and marginal cost (MC)

$$AC = \frac{TC}{Q} = \frac{Q^2}{Q} + \frac{8Q}{Q} + \frac{90}{Q} = Q + 8 + \frac{90}{Q}$$

$MC = \frac{dTC}{dQ} = 2Q + 8$

1.2.8 Revenue Function

If R is the total revenue of a firm, X is the quantity demanded or sold and P is the price per unit of output, we write the revenue function. Revenue function expresses revenue earned as a function of the price of good and quantity of goods sold. The revenue function is usually taken to be linear.

$$R = P \times X$$

Where R = revenue, P = price, X = quantity

If there are n products and $P_1, P_2 \ldots .. P_n$ are the prices and $X_1, X_2 \ldots \ldots X_n$ units of these products are sold then

$$R = P_1X_1 + P_2X_2 + \ldots \ldots \ldots + P_nX_n$$

Eg: $TR = 50 - 6Q^2$

Example: Given $P = Q^2 + 6Q + 5$, compute the TR function.

$TR = PQ = (Q^2 + 6Q + 5)Q = Q^3 + 6Q^2 + 5Q$. (Here quantity is assumed as Q)

1.2.9 Profit Function

Profit function as the difference between the total revenue and the total cost. If x is the quantity produced by a firm, R is the total revenue and C being the total cost then profit (π).

$P(x) = R(x) - C(x)$ or $\Pi = TR - TC$

Example: Given a $TR = 100Q - 5Q^2$ and $TC = Q^3 - 2Q^2 + 50Q$, find the profit function

$\Pi = TR - TC = (100Q - 5Q^2) - (Q^3 - 2Q^2 + 50Q)$

$= 100Q - 5Q^2 - Q^3 + 2Q^2 - 50Q = -Q^3 - 3Q^2 + 50Q$

1.2.10 Investment function

The investment function explains how the changes in national income induce changes in investment patterns in the national economy. It shows the functional relation between investment and the rate if interest or income. So, the investment function can be written as

$I = f(i)$, where, I is the investment and 'i' is the rate of interest.

MODULE II

MARGINAL CONCEPTS

Marginal utility, Marginal propensity to Consume, Marginal propensity to Save, Marginal product, Marginal Cost, Marginal Revenue, Marginal Rate of Substitution, Marginal Rate of Technical Substitution. Relationship between Average Revenue and Marginal Revenue- Relationship between Average Cost and Marginal Cost -Elasticity: Price elasticity, Income elasticity, Cross elasticity.

2.1 MARGINAL CONCEPTS

As you have already seen in your Microeconomics paper, marginal concepts relate to change in the total. For example, marginal utility is the change in total utility due to a change in the consumption. Marginal revenue is the change in total revenue due to a change in production. So marginal concepts refer to change and finding of a marginal function from a total function is basically a measurement of change. Since the derivative is a tool to measure change, we will see in this module how derivative is used to derive marginal functions from total functions. The simple rule to be followed is to find any marginal function from it total function, find the first order derivative of the total function.

2.1.1 Marginal Utility

Marginal utility is the addition made to total utility by consuming one more unit of commodity.

Marginal utility can be measured with the help of the following equation:

$$MU_{nth} = TU_n - TU_{n-1}$$

Or

$$MU = \frac{\Delta TU}{\Delta Q}$$

Given a total utility function for an individual consuming one commodity, TU = f(x), in terms of derivatives, marginal utility is

$$MU = \frac{dTU}{dx}$$

For an individual consuming more than one commodity with a totalutility function TU = f(x,y)

$$MU_x = \frac{\partial TU}{\partial x} \text{ and } MU_y = \frac{\partial TU}{\partial y}$$

Example:

1. Given a total utility function $U = x^2 + 3x + 5$, find marginal utility

$$MU = \frac{dTU}{dx} = 2x + 3$$

2. Given the total utility function U = 6xy + 9x + y, find marginal utility of x and y

$$MU_x = \frac{\partial u}{\partial x} = 6y - 9$$

$$MU_y = \frac{\partial u}{\partial y} = 6x + 1$$

2.1.2 Marginal Propensity to Consume

Marginal propensity to consume measures the change in consumption due to a change in income of the consumer. Mathematatiically, MPC is the first derivative of the consumption function.

Given a consumption function C = f(y), $MPC = \frac{dC}{dy}$

Example: Given a consumption function C = 100 + 0.5 Y, find MPC and MPS

$$MPC = \frac{dC}{dy} = 0.5$$

MPS = 1−MPC = 1 − 0.5 = 0.5

2.1.3 Marginal Propensity to Save

Marginal propensity to save measures the change in saving due to a change in income of the consumer. Mathematically, MPS is the first derivative of the saving function.

Given a consumption function S = f(y), $MPS = \frac{dS}{dy}$

Example: Given a saving function S = 80 + 0.4 Y find MPS and MPC

$$MPS = \frac{dS}{dy} = 0.4$$

MPC = 1−MPS = 1 − 0.4 = 0.6

2.1.4 Marginal Product

Marginal product of a factor of production refers to addition to total product due to the use of an additional unit of that factor.

The Marginal Product of Labour (MP_L) or Marginal Physical Product of Labour (MPP_L) is given by the change in TP due to a one unit change in the quantity of labour used. MP is derived by finding the derivative of the TP.

$MP_L = \frac{\partial TP}{\partial L}$ where L is labour, $MP_K = \frac{\partial TP}{\partial K}$ where K is capital

Example

Given a production function $Q = x^3 + 5xy + y^2$ for a firm which uses two inputs x and y in the production process, find marginal product of the two inputs.

$MPx = \frac{\partial Q}{\partial x} = 3x^2 + 5y$

$MPy = \frac{\partial Q}{\partial y} = 5x + 2y$

2.1.5 Marginal Cost

Marginal Cost (MC) refers to the change in total cost (TC) due to the production of an additional unit of output.

MC = $\frac{dTC}{dQ}$ where Q is output

Example: Given the total cost function TC = $x^2 - 4xy - 2y^3$, of a firm producing two goods x and y, find the marginal cost of x and y.

MCx = $\frac{\partial TC}{\partial x} = 2x - 4y$

MCy = $\frac{\partial TC}{\partial y} = -4x - 6y^2$

2.1.6 Marginal Revenue

Marginal Revenue (MR) is the change in total revenue (TR) due to the production of an additional unit of output.

MR = $\frac{\partial TR}{\partial Q}$ where Q is output

Example

Given the TR function of a firm producing two goods x and y, TR = $5xy^3 + 3x^2y$, find the marginal revenue from good x and good y.

MRx = $\frac{\partial TR}{\partial x} = 5y^3 + 6xy$

MRy = $\frac{\partial TR}{\partial y} = 15xy^2 + 3x^2$

2.1.7 Marginal Rate of Substitution (MRS)

As you have seen in the indifference curve analysis, marginal rate of substitution (MRS) for a consumer consuming two goods X and Y represents the rate at which the consumer is prepared to exchange goods X and Y. Thus for a consumer who uses two goods x and y, Marginal Rate of Substitution of x for y (MRSxy) is the amount of good y that the consumer is willing to give up to get one additional unit of good x. MRS is also referred to as RCS (Rate of Commodity Substitution). MRS thus refers to the change in the stock of good y due to a one unit change in the stock of good x. Therefore the derivativ $\frac{dy}{dx}$ measures MRS_{xy}.

MRS_{xy} is also equal to marginal utility of good x divided by marginal utility of good y.

$MRS_{xy} = \frac{MUx}{MUy}$

Example

1. Find MRS_{xy} for the function U = 12x + y

$$MR_{xy} = \frac{MUx}{MUy}$$

MUx = $\frac{\partial U}{\partial x} = 12$

MUy = $\frac{\partial U}{\partial y} = 1$

$$MR_{xy} = \frac{12}{1} = 12$$

2.1.8 Marginal Rate of Technical Substitution (MRTS)

Like MRS in the indifference curve analysis, you have seen MRTS in the iso-quant analysis.

MRTS represents the amount of one input the producer is willing to give up for obtaining an additional unit of the other input. This exchange or trade off will help the producer to stay on the same isoquant. Thus for a producer who uses two inputs K and L, Marginal Rate of Technical Substitution of L for K ($MRTS_{LK}$) is the amount of input Kthat the producer is willing to give up to get one additional unit of input L. The $MRTS_{LK}$ is also equal to $\frac{MP_L}{MP_K}$, the ratio of the marginal products of the two inputs. As the firm moves down an isoquant, $MRTS_{LK}$ diminishes.Note that MRTS is also equal to the negative of the slope of the isoquant.

Thus given the equation of an isoquant Q = ƒ (L, K) = c,

$MRTS_{LK} = -\frac{\partial K}{\partial L} = \frac{\partial Q/\partial L}{\partial Q/\partial K} = \frac{MP_L}{MP_K}$. Since $-\frac{\partial K}{\partial L}$ is the slope of an isoquant, MRTS is also equal to the negative of slope of an isoquant.

Example

1. Given an production function $Q = 6x^2 + 3xy + 2y^2$., find $MRTS_{xy}$ when y = 4 x = 5

$MRTS_{xy} = \frac{MP_x}{MP_y}$

$MP_x = \frac{\partial Q}{\partial x} = 12x + 3y$

$MP_y = \frac{\partial Q}{\partial y} = 3x + 4y$

at x = 5 and y = 4, MPx = 12(5) + 3(4) = 72

at x = 5 and y = 4, MPy = 3(5)) + 4(4) = 31

at x = 5 and y = 4, $MRTS_{xy} = \frac{MP_x}{MP_y} = \frac{72}{31} = 2.32$

2.1.9 Relationship between Average Revenue, Marginal Revenue

An important relationship between MR, AR (price) and price elasticity of demand which is extensively used in making price decisions by firms. This relationship can be proved algebraically also.

$$MR = P\left(1 - \frac{1}{e}\right)$$

Where P = Price and e = point elasticity of demand.

MR is defined as the first derivative of total revenue (TR).

Thus, MR = $\frac{\partial TR}{\partial Q}$ ……………………(1)

Now, TR is the product price and Q is the quantity of the product sold

(TR = P × Q),

Thus, MR = $\frac{\partial P.Q}{\partial Q}$

Using the product rule of differentiation, of a product, we have

$$MR = P\frac{dQ}{dQ} + Q\frac{dP}{dQ}$$

$$= P + Q\frac{dP}{dQ} \text{ ……………………….(2)}$$

This equation can be written as

$$MR = P\left(1 + \frac{Q}{P} \times \frac{dP}{dQ}\right) \text{……………..(3)}$$

Now, recall that point price elasticity of demand

$$= \frac{P}{Q} \times \frac{dQ}{dP}$$

It will thus be noticed that the expression

$\frac{P}{Q} \times \frac{dQ}{dP}$ in equation of the above is the reciprocal of point price elasticity of demand.

$\left(\frac{P}{Q} \times \frac{dQ}{dP}\right)$. Thus

$\frac{Q}{P} \times \frac{dP}{dQ} = \frac{1}{e}$ …………………… (4)

Substituting equation (4) in to equation (3), we obtain

$$MR = P\left(1 \pm \frac{1}{e}\right)$$

Or

$$= P\left(1 \frac{1}{e}\right)$$

Price or P is the same thing as average revenue (AR)

Therefore; $MR = AR\left(1 \frac{1}{e}\right)$

$$= AR\left(\frac{e-1}{e}\right)$$

Or

$$AR = MR\left(\frac{e}{e-1}\right)$$

2.1.10 Relationship between Average Cost and Marginal Cost

Average cost (AC) or Average total cost (ATC) is the cost of producing one unit of output. It is obtained by dividing total cost (TC) by quantity of output $\left(\frac{TC}{Q}\right)$. Marginal cost is the addition to total cost by the production of an additional unit of the commodity. MC is obtained by finding the first derivative of the TC function $\left(\frac{dTC}{dQ}\right)$. Since both AC and MC are obtained from the TC function, they are closely related.

The following may be noted regarding the relation between AC and MC.

- The slope of AC curve will be positive if and only if the marginal cost curve lies above the AC curve.

- The slope of AC curve will be zero if and only if the marginal cost curve interests the AC curve.

- The slope of AC curve will be negative if and only if the marginal cost curve lies below the AC curve.

2.2 ELASTICITY

Elasticity in common language is a measure of a variable's sensitivity to a change in another variable. In economics, as you have seen in Microeconomics course, elasticity refers the degree to which individuals, consumers or producers change their demand or the amount supplied in response to price or income changes. Elasticity can be found for changes in price of a good and response to it in quantity demanded of the good (price elasticity of demand), changes in price of a good and response to it in quantity supplied of the good (price elasticity of supply), changes in income and response to it in quantity of goods demanded (income elasticity of demand), changes in price of one good and response to it in quantity demanded of some other good – like its substitutes or complements (cross elasticity of demand) and so on.

2.2.1 Price Elasticity of Demand

The concept of elasticity is commonly used to assess the change in consumer demand as a result of a change in a good or service's price which is called price elasticity. Price elasticity of demand express the response of quantity demanded of a good to change in its price, given the consumer's income, his tastes and prices of all other goods.

Price elasticity can be found arithmetically as

$$\text{Price elasticity} = \frac{\text{proportionate change in quantity demanded}}{\text{Proportionate change in price}}$$

This equation in terms of calculus can be written as $\eta = \frac{dQ}{dP}\frac{P}{Q}$ where η is the coefficient of price elasticity of demand.

When $\eta > 1$, then demand is price elastic

When $\eta < 1$, then demand is price inelastic

When $\eta = 0$, demand is perfectly inelastic

When η = infinity, demand is perfectly elastic

Note that price elasticity of demand (Ep) is always negative, since the change in quantity demanded is in opposite direction to the change in price. But for the sake of convenience in understanding to the change in price, we ignore the negative sign and take in to account only the numerical value of the elasticity.

Example: Given the demand function q = −5p + 100, find price elasticity of demand when price is equal to 5.

$$\eta = \frac{dq}{dp}\frac{p}{q}$$

given q = −5p + 100, $\frac{dq}{dp} = -5$

when P = 5, Q = −5(5) + 100 = 75

substituting

$\eta = -5\ \frac{5}{75} = -0.33 = 0.33$ (since we discard the sign in measurement of price elasticity)

2.2.3 Income Elasticity

Income elasticity of demand shows the degree of responsiveness of quantity demanded of good to a small change in income of consumers. The degree of response of quantity demanded to a change in income is measured by dividing the proportionate change in quantity demanded by the proportionate change in income. Thus, more precisely, the income elasticity of demand may be defined as the ratio of the proportionate change in the quantity purchased of a good to the proportionate change in income which induce the former.

$$\text{Income elasticity} = \frac{\text{Proportinate change in purchased of a good}}{\text{proportionate change in income}}$$

In terms of calculus, if the function is given as Q = F(Y), $e_y = \frac{\partial Q}{\partial Y}\frac{Y}{Q}$

Based on the value of elasticity we can distinguish between different types of goods.

Normal Goods: Normal goods have a positive income elasticity of demand. So as income rise demand also rise for a normal good at each price level.

Necessary Goods: Necessities have an income elasticity of demand of between 0 and +1. Demand rises with income, but less than proportionately. Often this is because we have a limited need to consume additional quantities of necessary goods as our real living standards rise.

Luxuries: Luxuries have an income elasticity of demand greater than 1. (Demand rises more than proportionate to a change in income).

Inferior Goods: Inferior goods have a negative income elasticity of demand. Demand falls as income rises.

Example: Given Q = 700 – 2P + 0.02Y, find income elasticity of demand when P = 25 and Y = 5000.

$$e_y = \frac{\partial Q}{\partial Y}\frac{Y}{Q} = 0.02\left(\frac{5000}{750}\right) = 0.133$$

2.2.4 Cross elasticity of demand

Cross price elasticity (e_{xy}) measures the responsiveness of demand for good X due to a change in the price of good Y (a related good – could be a substitute or a complement or even an unrelated good).

e_{xy} = Proportionate change inthe demand for good X divided by Proportionate change inthe price for good Y

In terms of calculus, for a consumer using two goods x and y, the cross elasticity of demand may be written as $e_{xy} = \frac{\partial Q_x}{\partial P_y}\frac{P_y}{Q_x}$

With cross price elasticity we can make an important distinction between substitute products and complementary goods and services.

Substitutes: For substitute goods such as tea and coffee an increase in the price of one good will lead to an increase in demand for the other good. Cross price elasticity for two substitutes will be positive.

Complements: for complementary goods the cross elasticity of demand will be negative.

When there is no relationship between two goods, the cross price elasticity of demand is zero.

Example: Given $Q_1 = 100 - P_1 + 0.75P_2 - 0.25P_3 + 0.0075Y$. At $P_1 = 10$, $P_2 = 20$, $P_3 = 40$ and $Y = 10{,}000$, find the different cross elasticities of demand.

Here the given function relates the demand for a good (Q_1) and price of that good (P_1) and prices of other two related goods (P_2 and P_3).

Cross elasticity between good 1 and good 2 is given by $e_{12} = \frac{\partial Q_1}{\partial P_2}\frac{P_2}{Q_1}$

$$\frac{\partial Q_1}{\partial P_2} = 0.75$$

Substituting, $e_{12} = 0.75\,\frac{20}{170} = 0.088$

since e_{12} is positive, good 1 and 2 are substitutes. An increase in P_2 will lead to an increase in Q_1.

Cross elasticity between good 1 and good 3 is given by $e_{13} = \frac{\partial Q_1}{\partial P_3}\frac{P_3}{Q_1}$

$$\frac{\partial Q_1}{\partial P_3} = -0.25$$

$$e_{13} = -0.25\,\frac{40}{170} = -\,0.059$$

since e_{13} is negative, good 1 and 3 are compliments. An increase in P_3 will lead to a decrease in Q_1.

Exercises:

1. If the demand Law is given by $q = \frac{20}{p+1}$, find the elasticity of demand with respect to price at the point when p = 3.

Solution: Elasticity of demand $= -\,\frac{\Delta q}{\Delta p} \times \frac{p}{q}$

$$q = 20\,(p + 1)^{-1}$$

$\frac{\Delta q}{\Delta p} = -\,20\,(p + 1)^{-2}$

$= -\frac{-20}{(p+1)2}$

When p = 3,

q = 20/4

= 5

$\frac{\Delta q}{\Delta p}$ = - 20/16

= - 5/4

Elasticity of demand = 5/4 × 3/5

= ¾

2. The demand function p = 50 – 3x, when p = 5, then 5 = 50 – 3x

X = 15

$\frac{d}{dp}(p) = \frac{d}{dp}(50 - 3x)$

$1 = -3\frac{dx}{dp}$

or

$\frac{dx}{dp}$ = - 1/3

$e = \frac{dx}{dp} \times \frac{p}{x}$

= 5/15 × 1/3

= 1/9

3. The total cost C (x) associated with producing and marketing x units of an item is given by C(x) = .005 x^3 - .0 x² - 30ox' +300. Find

(1) Total cost when output is 4 units.
(2) Average cost of output of 10 units.

Solution: (1) Given that C(x) = .005 x^3 - .02 x² - 30x + 3000.

For x = 4 units, the that cost C(x) becomes

C(x) = $.005(4)^3 - .02\ (4)^2 - 30x \times 4 + 3000$

= .32 - .32 – 120 + 3000

= Rs. 2880

(2) Average cost (AC) = TC/x

$= \frac{.005\,x^3 - .02x^2 - 30x + 3000}{x}$

$= .005\,x^2 - .02x - 30$

4. A function p = 50 – 3x, find TR, AR, MR

TR = P × x

= (50 – 3x) x

= $50x - 3x^2$

AR = TR/x

= $50x - 3x^2$

= 50 – 3x

MR = $\frac{d(50x-3x^2)}{dx}$

= $\frac{d(50x)}{dx}$ - $\frac{d(3x^2)}{dx}$

= 50x – 6x

5. The demand function for mutton is:

Q_M = 4850 - $5P_M$ + $1.5P_0$ + .1Y

Find the income elasticity of demand and cross elasticity of demand for mutton. Y (income) = Rs.1000 P_M(price of mutton) = Rs.200,P_0(price of chicken) Rs.100.

Solution: income elasticity = $\frac{dq}{dy} \times \frac{Y}{Q}$

$\frac{dq}{dy}$ = .1 income elasticity = (.1) × 1000

= 4850 – 1000 + 150 + 100

= 5100 – 1000

= 4100

Income elasticity = $\frac{.1 \times 1000}{4100}$

= $\frac{1}{41}$

Cross elasticity of mutton = $dq_m / dp_0 \times p_0 / q_m$

= dq_m / dp_0

= 1.5, cross elasticity

= 1.5 × $\frac{100}{4100}$

= $\frac{1.5}{41}$

= $\frac{3}{2 \times 41}$

= $\frac{3}{82}$

8. Find the elasticity of supply for supply function x = $2p^2$+ 5 when p = 3

$Es = \frac{p}{x} \times \frac{dx}{dp}$

$= \frac{p}{x} \times 4p$

$= \frac{4 \times 9}{18+5}$

$= \frac{36}{23}$

9. For a firm, given that c = 100 + 15x and p = 3 .find profit function

Profit function π = TR –TC

TR = p × x

= 3 x

Π = 3x – 100 + 15x

= 18x -100

10. The demand function p = 50 – 3x. Find MR

TR = P × X

= (50 – 3x) x

$= 50x - 3x^2$

$MR = \frac{dTR}{dX}$

= 50 – 6x

12. The total cost function is $TC = 60 - 12x + 2x^2$. Find the MC

$MC = \frac{dTC}{dx}$

= 12 + 4x

MODULE III

OPTIMISATION

Optimisation of single / multi variable functions - Constrained optimisation with Lagrange Multiplier –Significance of Lagrange Multiplier. Economic applications: Utility Maximisation, Cost Minimisation, Profit Maximisation.

3.1 Optimization of Functions

Optimization is the process of finding relative maximum or minimum of a function. There are two conditions for obtaining optimum value of a function.

1. The First order conditions: Set the first derivative equal to zero and solve for the critical values. It identifies all the points at which function is neither increasing nor decreasing, but at a plateau.
2. Take the second derivative, evaluate it at critical point(s) and check the sign.
   f(a) < 0 for maximum
   And f (a)>0 for minimum.

The problem of optimization of some quantity subject to certain restrictions or constraint is a common feature of economics, industry, defence, etc. The usual method of maximizing or minimizing a function involves constraints in the form if equations. Thus utility may be maximized subject to the budget constraint of fixed income, given in the form of equation. The minimization of cost is a familiar problem to be solved subject to some minimum standards. If the constraints are in the form of equations, methods of calculus can be useful. However, if the constraints are inequalities instead of equations and we have an objective function may be optimized subject to these inequalities, we use the method of mathematical programming.

3.1.2 Substitution Method

Another method of solving the objective function with subject to the constraint is substitution methods. In this method, substitute the values of x or y, and the substitute this value in the original problem, differentiate this with x and y.

Consider a utility function of a consumer

$$U = x^{0.3} + y^{0.3} ,$$

The budget constraint 20x + 10y = 200.

Rewrite the above equation $y = \frac{200-20x}{10}$

$$= 20 - 2x$$

Then the original utility function U = $x^{0.3} + (20 - 2x)$

3.1.3: Lagrange Method

Constrained maxima and minima: in mathematical and economic problems, some relation or constraint sometimes restricts the variables in a function. When we wish to maximize or minimize f (x_1, x_2…x_n) subject to the condition or constraint g(x_1, x_2 …x_n) = 0, there exist a method known as the method of Lagrange Multiplier. For example utility function U = u(x_1, x_2…x_n) may be subject to the budget constraint that income equals expenditure that is Y = $p_1x_1 + p_2x_2 + \ldots\ldots + p_nx_n$. We introduce a new variable γ called the Lagrange Multiplier and construct the function.

$$Z = f(x_1, x_2 \ldots x_n) + \gamma \text{ g } g(x_1, x_2 \ldots x_n)$$

This new function z is a function on n + 1 variable $x_1, x_2 \ldots x_n$ and γ

Example: Given the utility function U = $4{x_1}^{1/2}{x_2}^{1/2}$ an budget constraint 60 = $2x_1 + x_2$. Find the condition for optimality.

(Hint. U = $4{x_1}^{1/2}{x_2}^{1/2}$

$$F(x_1, x_2, \gamma) = 4{x_1}^{1/2}{x_2}^{1/2} + \gamma\,(60 - 2x_1 - x_2)$$

$$\frac{\partial F}{\partial x_1} = 4{x_2}^{1/2} \,.\, 1/2{x_1}^{-1/2} - 2\gamma = 0$$

$$\frac{\partial F}{\partial x_2} = 4{x_1}^{1/2} \,.\, 1/2{x_2}^{-1/2} - \gamma = 0$$

$$\frac{\partial F}{\partial \gamma} = 60 - 2x_1 - x_2 = 0$$

$$= 4{x_2}^{1/2}.1/2\ {x_1}^{-1/2} / 4{x_1}^{1/2}.1/2\ {x_2}^{-1/2}$$

$$= \frac{2\gamma}{\gamma}$$

$$= \frac{x_2}{x_1}$$

$$= \frac{2}{1}$$

Or

$= 2x_2$

$= x_2$

3.2 Utility Maximization

There are two approaches to study consumer behaviour- the first approach is a classical one and is known as cardinal utility approach and the second approach is ordinal utility approach popularly known as indifference curve approach. In both the approaches, we assume that consumer always behaves in a rational manner, because he derives the maximum utility (satisfaction) out of his budget constraint.

Example: The utility function of the consumer is given by u $= x_1 x_2^2 - 10x_1$ where x_1 and x_2 are the quantities of two commodities consumed. Find the optimal utility value if his income is 116 and product prices are 2 and 8 respectively.

Solution: we have utility function

$$U = f(x_1, x_2) = u = x_1 x_2^2 - 10x_1, \text{ and}$$

Budget constraint $116 - 2x_1 - 8x_2 = 0$ from budget equation we get $x_1 = 58 - 4x_2$

$$U = (58 - 4x_2)x_2^2 - 10(58 - 4x_2)$$

$$= 58x_2^2 - 4x_2^3 - 580 - 40x_2$$

For minimum utility

$$\frac{du}{dx_2} = 0$$

$$= 116x_2 - 12x_2^2 - 40 = 0$$

$$3x_2^2 - 29x_2 - 10 = 0$$

$$3x_2^2 - 30x_2 + x_2 - 10 = 0$$

$$3x_2(x_2 - 10) + (x_2 - 10) = 0$$

$$(3x_2 + 1)(X_2 - 10) = 0$$

Or

$x_2 = -1/3$

Or

$x_2 = 10x_2$ cannot be negative.

Hence $x_2 = 10$

$$\frac{d^2u}{dx^2} = 116 - 24x_2$$

$$= 116 - 240$$

$= -124 < 0$ maxima is confirmed.

$x_2 = 10$,

$x_1 = 58 - 40$

$$= 18$$

We know that consumer's equilibrium (condition of maximum utility) fulfilled when

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

$$MU_1 = x_2^2 - 10$$

$$MU_2 = 2x_2x_1$$

$$\frac{MU_1}{MU_2} = \frac{x^{22} - 10}{2x_2x_1}$$

$$= \frac{100-10}{2 \times 18 \times 10}$$

$$= \frac{90}{360}$$

$$= \frac{1}{4}$$

Hence maximum utility is obtained when $x_1 = 18$ and $x_2 = 10$

3.3 Cost Minimization

Cost minimization involves how a firm has to produce a given level of output with minimum cost. Consider a firm that uses labour (L) and capital (K) to produce output (Q). Let W is the price of labour, that is, wage rate and r is the price of capital and the cost (C) incurred to produce a level of output is given by

$$C = wL + rK$$

The objective of the form is to minimize cost for producing a given level of output. Let the production function is given by following.

$$Q = f(L, K)$$

In general there is several labour – capital combinations to produce a given level of output. Which combination of factors a firm should choose which will minimize its total cost of production. Thus, the problem of constrained minimization is

$$\text{Minimize } C = wL + rK$$

Subject to produce a given level of output, say Q_1 that satisfies the following production function

$$Q_1 = f(L, K)$$

The choice of an optimal factor combination can be obtained by using Lagrange method.

Let us first form the Lagrange function is given below

$$Z = wL + rK + \gamma(Q_1 - f(L, K))$$

Where γ is the lagrange multiplier

For minimization of cost it necessary that partial derivatives of Z with respect to L, K and γ be zero

$\frac{\partial Z}{\partial L} = w - \frac{\gamma \partial f(L,K)}{\partial L} = 0$

$\frac{\gamma \partial Z}{\partial K} = r - \frac{\gamma \partial f(L,K)}{\partial K} = 0$

$\frac{\partial Z}{\partial \gamma} = Q_1 - f(L, K) = 0$

Note that $\frac{\partial f(L,K)}{\partial L}$ and $\frac{\partial f(L,K)}{\partial K}$ are the marginal physical products of labour and capital respectively.

Rewriting the above equation we have

$$w - \gamma MP_L = 0$$

$$r - \gamma MP_K = 0$$

$$Q_1 = f(L, K)$$

By combining the two equations, we have

$\frac{w}{r} = MP_L / MP_K$

The last equations shows that total cost is minimized when the factor price ratio $\frac{w}{r}$ equal the ratio of MPP of labour and capital.

3.4: Profit Maximization

Maximizations of profit subject to the constraint can also used to identify the optimum solution for a function.

Assumes that TR = PQ

$$TC = wL + rK$$

$$\Pi = TR - TC$$

$$\Pi = PQ - (wL + rK)$$

Thus the objective function of the firm is to maximize the profit function

$$\Pi = PQ - (wL + rK)$$

The firm has to face a constraint Q = f (K, L)

From the Lagrange function,

$$Z = (PQ - (wL + rK)) + \gamma(F(K, L) - Q)$$

$\frac{\partial Z}{\partial Q} = P - \gamma = 0$.....................(1)

$\frac{\partial Z}{\partial L} = -w + \gamma fL = 0$…...............(2)

$\frac{\partial Z}{\partial K} = -r + \gamma fK = 0$… ....(3)

$\frac{\partial Z}{\partial Q} = f(K, L) - Q = 0$…......(4)

From equation (1), we get $P = \gamma$

Substitute this in (2) and (3)

From (2)

$$w = \gamma f L$$

Substituting $P = \gamma$

We get $w = PfL$ ....(5)

Equation (3)

$r = PfK$ …… (6)

Now in (5) $P = w/fL$

And in (6) $P = r/fK$ rewrite the above equation

$$W/fL = r/fK$$

Cross multiplying $w/r = fL/fK$

The above condition is profit maximization

MODULE IV
PRODUCTION FUNCTION, LINEAR PROGRAMMING AND INPUT OUTPUT ANALYSIS

Production function- homogeneous and non-homogeneous - Degree of homogeneity and returns to scale – Properties of Cobb-Douglas production function. - Production possibility curve. - Linear programming: Basic concept, Nature of feasible, basic and optimal solution; Graphic solution –The Dual - Applications of linear programming in economics. Input-output analysis –Matrix of technical coefficients –The Leontief matrix –computation of total demand for a two/ three sector economy.

4.1 PRODUCTION FUNCTION

Production function is a transformation of physical inputs in to physical out puts. The output is thus a function of inputs. The functional relationship between physical inputs and physical output of a firm is known as production function. Algebraically, production function can be written as,

$$Q = f(a,b,c,d,\ldots..)$$

Where, Q stands for the quantity of output, a,b,c,d, etc; stands for the quantitative factors. This function shows that the quantity (q) of output produced depends upon the quantities, a, b, c, d of the factors A, B, C, D respectively.

The general mathematical form of Production function is:

$$Q = f(L,K,R,S,v,e)$$

Where: Q stands for the quantity of output, L is the labour, K is capital, R is raw material, S is the Land, v is the return to scale and e is efficiency parameter.

According to G.J. Stigler, "the production function is the name given the relationship between the rates of inputs of productive services and the rates of output of product. It is the economists summary of technological knowledge. Thus, production function express the relationship between the quantity of output and the quantity of various input used for the production. More precisely the production function states the maximum quantity of output that can be produced from any given quantities of various inputs or in other words, if stands the minimum quantities of various inputs that are required to yield a given quantities of output.

"Production function of the firm may also be derived as the minimum quantities of wood, varnish, labour time, machine time, floor space, etc; that are required to produce a given number of table per day".

Knowledge of the production function is a technological or engineering knowledge and provided to the form by its engineers or production managers. Two things must be noted that in respect of production function. First, production functions like demand function, must be considered with reference to a particular period of time. Production function expresses flows of inputs resulting in flows of output in a specific period of time. Secondly, production function of a firm is determined by the state of technology. When there is advancement in technology, the production function charges with the result that the new production function charges with the result of output from the given inputs, or smaller quantities of inputs can be used for producing a given quantity of output.

4.1.1 Linear, Homogeneous Production Function

Production function can take several forms but a particular form of production function enjoys wide popularity among the economists. This is a linear homogeneous production function, that is, production function which is homogenous production function of the first degree. Homogeneous production function of the first degree implies that if all factors of production are increased in a given proportion, output also increased in a same proportion. Hence linear homogeneous production function represents the case of constant return to scales. If there are two factors X and Y, The production function and homogeneous production function of the first degree can be mathematically expressed as,

$Q = f(X, Y)$ Where Q stands for the total production, X and Y represent total inputs.

$mQ = f(mX, mY)$ m stands any real number

The above function means that if factors X and Y are increased by m-times, total production Q also increases by m-times. It is because of this that homogeneous function of the first degree yield constant return to scale.

More generally, a homogeneous production function can be expressed as

$Qmk = (mX,mY)$

Where m is any real number and k is constant. This function is homogeneous function of the k^{th} degree. If k is equal to one, then the above homogeneous function becomes homogeneous of the first degree. If k is equal to two, the function becomes homogeneous of the 2^{nd} degree.

If $k>1$, the production function will yield increasing return to scale.

If $k<1$, it will yield decreasing return to scale.

4.1.2 Fixed Proportion Production Function

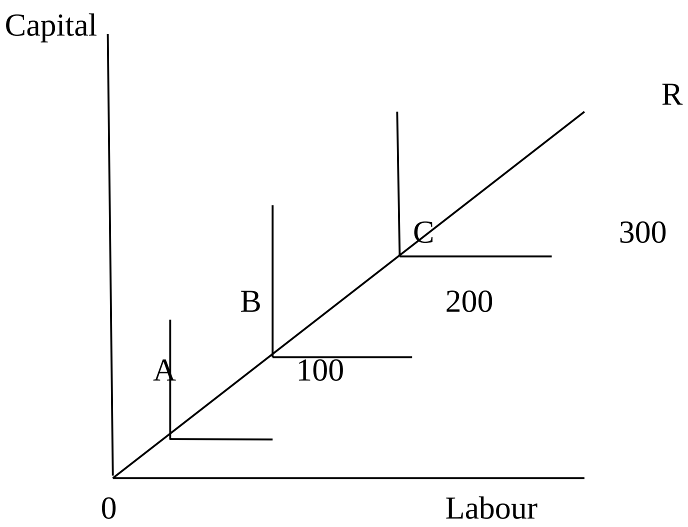
Production function is of two qualitatively different forms. It may be either fixed-proportion production function or variable proportion production functions. Whether production function is of a fixed proportion form or a variable proportion form depends upon whether technical coefficients of production are fixed or variable. The amount of a productive factor that is essential to produce a unit of product is called the technical coefficient of production. For instance, if 25 workers are required to produce 100 units of a product, then 0.25 is the technical coefficient of labour for production. Now, if the technical coefficient of production of labour is fixed, then 0.25 of labour unit must be used for producing a unit of product and its amount cannot be reduced by using in its place some other factor. Therefore, in case of fixed proportions production function, the factor or inputs, say labour and capital, must be used in a definite fixed proportion in order to produce a given level of output. A fixed proportion production function can also be illustrated by equal product curve or isoquants. As in fixed proportion production function, the two factors, say capital and labour, must be used in fixed ratio, the isoquants of such a production function are right angled.

Suppose in the production of a commodity, capital- labour ratio that must be used to produce 100 units of output is 2:3. In this case, if with 2 units of capital, 4 units of labour are used, then extra one unit of labour would be wasted; it will not add to total output. The capital-labour ratio must be maintained whatever the level of output.

If 200 units of output are required to be produced, then, given the capital- output ratio of 2:3, 4 units of capital and 6 units of labour will have to be used.

If 300 units of output are to be produced, then 6 units of labour and 9 units labour will have to be used.

Given the capital –labour ratio of 2:3, an isoquant map of fixed –proportion production function has been drawn in the given figure.



In a fixed proportion production function, doubling the quantities of capital and labour at the required ratio doubles the output, trebling their quantities at the required ratio trebles the output.

4.1.3 Cobb – Douglas Production Function

Many Economists have studied actual production function and have used statistical methods to find out relations between changes in physical inputs and physical outputs. A most familiar empirical production function found out by statistical methods is the Cobb – Douglas production function. Cobb – Douglas production function was developed by Charles Cobb and Paul Douglas. In C-D production function, there are two inputs, labour and capital, Cobb – Douglas production function takes the following mathematical form

$$Q = AL^{\alpha}K^{\beta}$$

Where Q is the manufacturing output, L is the quantity of labour employed, K is the quantity of capital employed, A is the total factor productivity or technology are assumed to be a constant. The α and β, output elasticity's of Labour and Capital and the A,α and β are positive constant.

Roughly speaking, Cobb –Douglas production function found that about 75% of the increasing in manufacturing production was due to the Labour input and the remaining 25 % was due to the Capital input.

4.1.4 Properties ofCobb – Douglas Production Function

1. Average product of factors: The first important properties of C – D production function as well as of other linearly homogeneous production function is the average and marginal products of factors depend upon the ratio of factors are combined for the production of a commodity . Average product if Labour (APL) can be obtained by dividing the production function by the amount of Labour L. Thus,

Average Product Labour (Q/L)

$$Q = AL^{\alpha}K^{\beta}$$

$$Q/L = AL^{\alpha}K^{\beta} \;/ \text{L}$$

$$= AK^{\beta} \;/L^{1-\alpha}$$

$$= \text{A}(K/L)^{\beta}$$

Thus Average Product of Labour depends on the ratio of the factors (K/L) and does not depend upon the absolute quantities of the factors used.

Average Product of Capital (Q/K)

$$Q = AL^{\alpha}K^{\beta}$$

$$Q/K = AL^{\alpha}K^{\beta} \;/ \text{K}$$

$$= AL^{\alpha}K^{\beta} \;/ \text{K}$$

$$= AL^{\alpha} \;/K^{1-\beta}$$

$$= \text{A}(\frac{L}{K})^{\alpha}$$

So the average Product of capital depends on the ratio of the factors (L/K) and does not depend upon the absolute quantities of the factors used.

2 Marginal Product of Factors: The marginal product of factors of a linear homogenous production function also depends upon the ratio of the factors and is independent of the absolute quantities of the factors used. Note, that marginal product of factors, says Labour, is the derivative of the production function with respect to Labour.

$$Q = AL^{\alpha}K^{\beta}$$

$$\mathbf{MP_L} = \frac{\partial Q}{\partial L}$$

$= \alpha A L^{\alpha-1} K^{\beta}$

$= A\alpha K^{\beta} / L^{1-\alpha}$

$= \alpha A (\frac{K}{L})^{\beta}$

$MP_L = \alpha AP_L$

It is thus clear that MP_L depends on capital –labour ratio, that is, Capital per worker and is independent of the magnitudes of the factors employed.

$Q = AL^{\alpha} K^{\beta}$

$MP_K = \frac{\partial Q}{\partial K}$

$= \beta K^{\beta-1} L^{\alpha}$

$= \beta AL^{\alpha} / K^{1-\beta}$

$= \beta A (\frac{L}{K})^{\alpha}$

$MP_L = \beta AP_K$

It is thus clear that MPL depends on capital – labour ratio, that is, capital per worker and is independent of the magnitudes of the factors employed.

$Q = AL^{\alpha} K^{\beta}$

$MP_K = \frac{\partial Q}{\partial K}$

$= \beta K^{\beta-1} AL^{\alpha}$

$= \beta AL^{\alpha} / K^{1-\beta}$

$= \beta A (\frac{L}{K})^{\alpha}$

$= \beta AP_K$

3 Marginal rate of substitution: Marginal rate of substitution between factors is equal to the ratio of the marginal physical products of the factors. Therefore, in order to derive MRS from Cobb –Douglas production function, we used to obtain the marginal physical products of the two factors from the C – D function.

$Q = AL^{\alpha} K^{\beta}$

Differentiating this with respect to L, we have

$MP_L = \frac{\partial Q}{\partial L}$

$= \partial AL^{\alpha} K^{\beta}$

$= \alpha AL^{\alpha-1} K^{\beta}$

$= \frac{\alpha(AL^{\alpha} K^{\beta})}{L^{1}}$

Now, $Q = AL^{\alpha} K^{\beta}$, Therefore,

$= \alpha (\frac{Q}{L})$

$\frac{dQ}{dL}$ Represents the marginal product of labour and $\frac{Q}{L}$ stands for the average of labour.

Thus, $MP_L = \alpha(AP_L)$

Similarly, by differentiating C –D production function with respect to capital, we can show that marginal product of capital

$$Q = AL^{\alpha}K^{\beta}$$

$$MP_K = \frac{\partial Q}{\partial K}$$

$$= \beta AL^{\alpha}K^{\beta-1}$$

$$= \frac{\beta(AL^{\alpha}K^{\beta})}{K^1}$$

$$= \beta(\frac{Q}{K})$$

$$MRS_{LK} = \frac{MPL}{MPK}$$

$$= \frac{\alpha(\frac{Q}{L})}{\beta(\frac{Q}{K})}$$

$$= \frac{\alpha}{\beta} \times \frac{K}{L}$$

4.C –D production function and Elasticity of substitution (ℓs or σ) is equal to unity.

$$ℓs = \frac{Proportionate change in Capital-Labour ratio(\frac{K}{L})}{Proportionate change in MRSLK}$$

$$= \frac{d(\frac{K}{L})}{K/L} \Big/ d(\frac{MRSLK}{MRSLK})$$

Substituting the value of MRS obtain in above

$$= \frac{\frac{d(\frac{K}{L})}{K/L}}{\frac{\partial(\frac{\alpha}{\beta}\times\frac{K}{L})}{\beta} \hat{} K/L} = \frac{\frac{d(\frac{K}{L})}{K/L}}{\frac{\partial(\frac{\alpha}{\beta}\times\frac{K}{L})}{\beta} \hat{} K/L} = 1$$

5. Return to Scale: An important property of C –D production function is that the sum if its exponents measures returns to scale. That is, when the sum of exponents is not necessarily equal to zero is given below.

$$Q = AL^{\alpha}K^{\beta}$$

In this production function the sum of exponents (α + β) measures return to scale. Multiplying each input labour (L) and capital (K), by a constant factor g, we have

$$Q' = A(gL)^{\alpha}(gK)^{\beta}$$

$$= g^{\alpha}g^{\beta}(AL^{\alpha}K^{\beta} \quad \text{ie} \quad a^m \times a^n = a^{m+n}$$

$$= g^{\alpha+\beta}(AL^{\alpha}K^{\beta})$$

i.e. Q' = $g^{\alpha+\beta}$ Q

This means that when each input is increased by a constant factor g, output Q increases by$g^{\alpha+\beta}$. If α + β = 1 then, in this production function.

$$Q' = g'Q$$

$$Q' = gQ$$

This is, when α + β = 1, output (Q) also increases by the same factor g by which both inputs are increased. This implies that production function is homogeneous of first degree or, in other words, retune to scale are constant.

When α + β > 1, say it is equal to 2, then, in this production function new output.

$$Q' = g^{\alpha+\beta} AL^{\alpha}K^{\beta}$$

$$= g^2Q.$$

In this case multiplying each input by constant g, then output (Q) increases by g^2. Therefore, α + β > 1.

C –D production function exhibits increasing return to scale. When α + β < 1, say it is equal to 0.8, then in this production function, new output.

$$Q' = g^{\alpha+\beta} AL^{\alpha}K^{\beta}$$

$$= g^{0.8}Q.$$

That is increasing each input by constant factor g will cause output to increase by$g^{0.8}$, that is ,less than g .Return to scale in this case are decreasing. Therefore α + β measures return to scale.

If α + β = 1, return to scale are constant.

If α + β > 1, return to scale are increasing.

If α + β < 1, return to scale are decreasing.

4.1.5 C-D Production Functions and Output Elasticity of Factors

The exponents of labour and capital in C –D production function measures output elasticity's of labour and capital. Output elasticity of a factor refers to the relative or percentage change in output caused by a given percentage change in a variable factor, other factors and inputs remaining constant. Thus,

$$O\,E = \frac{\partial Q}{\partial L} \times \frac{L}{Q}$$

$$= a \times \frac{Q}{L} \times \frac{L}{Q} = a.$$

Thus, exponent (a) of labour in C –D production function is equal to the output elasticity of labour.

Similarly, O E of Capital = $\frac{\partial Q}{\partial K} \times \frac{K}{Q}$

MPk = b.$\frac{K}{Q}$

$= b.\frac{Q}{K} \times \frac{K}{Q}$

$= b$

Therefore, output elasticity of capital = $b.\frac{Q}{K} \times \frac{K}{Q}$

$= b$

4.1.6: C–D production Function and Euler's theorem

C–D production function $Q = AL^{\alpha}K^{\beta}$

Where a + b = 1 helps to prove Euler theorem. According to Euler theorem, total output Q is exhausted by the distributive shares of all factors .when each factor is paid equal to its marginal physical product. As we know

$MP_L = A\ a(\frac{K}{L})^b$

$MP_K = A\ a\ (\frac{L}{K})^a$

According to Euler's theorem if production functions is homogeneous of first degree then, Total output, Q = L × MP_L + K × MP_K, substituting the values of MP_L and MP_K, we have

$Q = L \times Aa(\frac{K}{L})^b + K \times A\ b(L/K)^a$

$= AaL^{1-b}K^b + AbL^aK^{1-a}$

Now, in C – D production function with constant return to scale a + b = 1 and

Therefore: a = 1 – b and b = 1 – a, we have

$Q = AaL^aK^b + Ab\ L^aK^b$

$= (a + b)\ A\ L^aK^b$

Since a + b = 1 we have

$Q = A\ L^aK^b$

Q = Q

Thus, in C – D production function with a + b == 1 if wage rate = MP_L and rate of return on capital (K) = MP_K, then total output will be exhausted.

4.1.7: C–D Production Function and Labour Share in National Income.

C–D production function has been used to explain labour share in national income (i.e., real national product). Let Y stand for real national product, L and K for inputs of labour and capital, then according to C – D production function as applied to the whole economy , we have

Y =..AL^aK^{1-a} ……….. (1)

Now, the real wage of labour (w) is its real marginal product. If we differentiate Y partially with respect to L, we get the marginal product of labour, thus, Real wage (or marginal product of labour)

$$W = \frac{dY}{dL}$$

$$= a\ AL^{a-1}K^{1-a} \quad \ldots\ldots\ldots\ldots(2)$$

$$\text{Total wage bill} = wL = \frac{dY}{dL}$$

$$= a\ AL^{a}K^{1-a}$$

From (1) and (2), we get,

The labour share in real national product

$$= \frac{Total\ wage\ bill}{Real\ national\ product}$$

$$= \frac{wL}{Y}$$

$$= \frac{a\ AL^{a}K^{1-a}}{AL^{a}K^{1-a}}$$

$$= a$$

Thus, according to C – D production function, labour's share in real national product will be a constant 'a' which is independent of the size of labour force.

4.2 LINEAR PROGRAMMING PROBLEMS (LPP)

The term linear programming consists of two words, linear and programming. The linear programming considers only linear relationship between two or more variables. By linear relationship we mean that relations between the variable can be represented by straight lines. Programming means planning or decision- making in a systematic way. "Linear programming refers to a technique for the formulation and solution of problems in which some linear function of two or more variables is to be optimized subject to a set of linear constraints at least one of which must be expressed as inequality". American mathematician George B. Danzig, who invented the linear programming technique.

Linear programming is a practical tool of analysis which yields the optimum solution for the linear objective function subject to the constraints in the form of linear inequalities. Linear objective function and linear inequalities and the techniques, we use is called linear programming, a special case of mathematical programming.

4.2.1 Terms of Linear Programming

(1) Objective Function

Objective function, also called criterion function, describe the determinants of the quantity to be maximized or to be minimized. If the objective of a firm is to maximize output or profit, then this is the objective function of the firm. If the linear programming requires the minimization of cost, then this is the objective function of the firm. An objective function has two parts – the primal and dual. If the primal of the objective function is to maximize output then its dual will be the minimization of cost.

(2) Technical Constraints

The maximization of the objective function is subject to certain limitations, which are called constraints. Constraints are also called inequalities because they are generally expressed in the form of inequalities. Technical constraints are set by the state of technology and the availability of factors of production. The number of technical constraints in a linear programming problem is equal to the number of factors involved it.

(3) Non- Negativity Constraints

This express the level of production of the commodity cannot be negative, ie it is either positive or zero.

(4) Feasible Solutions

After knowing the constraints, feasible solutions of the problem for a consumer, a particular, a firm or an economy can be ascertained. Feasible solutions are those which meet or satisfy the constraints of the problem and therefore it is possible to attain them.

(5) Optimum Solution

The best of all feasible solutions is the optimum solution. In other words, of all the feasible solutions, the solution which maximizes or minimizes the objective function is the optimum solution. For instance, if the objective function is to maximize profits from the production of two goods, then the optimum solution will be that combination of two products that will maximizes the profits for the firm. Similarly, if the objective function is to minimize cost by the choice of a process or combination of processes, then the process or a combination of processes which actually minimizes the cost will represent the optimum solution. It is worthwhile to repeat that optimum solution must lie within the region of feasible solutions.

4.2.2 Assumptions of LPP

The LPP are solved on the basis of some assumptions which follow from the nature of the problem.

(a) Linearity

The objective function to be optimized and the constraints involve only linear relations. They should be linear in their variables. If they are not, alternative technique to solve the problem has to be found. Linearity implies proportionality between activity levels and resources. Constraints are rules governing the process.

(b) Non- negativity

The decision variable should necessarily be non –negative.

(c) Additive and divisibility

Resources and activities must be additive and divisible.

(d) Alternatives

There should be alternative choice of action with a well defined objective function to be maximized or minimized.

(e) Finiteness

Activities, resources, constraints should be finite and known.

(f) Certainty

Prices and various coefficients should be known with certainty.

4.2.3 Application of linear programming

There is a wide variety of problem to which linear programming methods have been successfully applied.

- **Diet problems**

To determine the minimum requirements of nutrients subjects to availability of foods and their prices.

- **Transportation problem**

To decide the routes, number of units, the choice of factories, so that tha cost of operation is the minimum.

- **Manufacturing problems**

To find the number of items of each type that should be made so as to maximize the profits.

- **Production problems**

Subject to the sales fluctuations. To decide the production schedule to satisfy demand and minimize cost in the face of fluctuating rates and storage expenses.

- **Assembling problems**

To have, the best combination of basic components to produce goods according to certain specifications.

- **Purchasing problems**

To have the least cost objective in, say, the processing of goods purchased from outside and varying in quantity, quality and prices.

- **Job assigning problem**

To assign jobs to workers for maximum effectiveness and optimum results subject to restrictions of wages and other costs.

2.2.4 Limitations of LPP

The computations required in complex problems may be enormous. The assumption of divisibility of resources may often be not true. Linearity of the objective function and constraints may not be a valid assumption. In practice work there can be several objectives, not just a single objective as assumed in LP.

4.2.5 Formulation of Linear Programming

The formulation has to be done in an appropriate form. We should have,

(1) An objective function to be maximized or minimized. It will have n decision variables x_1, x_2 ….x_n and is written in the form.

$$\text{Max (Z) or Min(c)} = C_1X_1 + C_2X_2 + \ldots\ldots\ldots\ldots + C_nX_n$$

Where, each *Cj* is a constant which stands for per unit contribution of profit (in the maximization case) or cost (in the minimization case to each *Xj*)

(2) The constraints in the form of linear inequalities.

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{1n}x_n < \text{ or } > b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n < \text{ or } > b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n < \text{ or } > b_n$$

Briefly written

$$\sum a_{ij}x_j < \text{ or } > b_i \qquad I = 1,2,\ldots\ldots n$$

Where b_i, stands for the i^{th} requirement or constraint

The non-negativity constraints are

$$x_1, x_2, \ldots\ldots x_n \geq 0$$

In matrix notation, we write

$$\text{Max or Min } Z = CX$$

Subject to constraints $AX \leq b$ or $AX \geq b$ and the non-negativity conditions $x \geq 0$

Where $C = [\, c_1, c_2, \ldots..c_n \,]$ here

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ \ldots & - & - & - \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$$

Examples: A firm can produce a good either by (1) a labour intensive technique, using 8 units of labour and 1 unit of capital or (2) a capital intensive technique using 1 unit of labour and 2 unit of capital. The firm can arrange up to 200 units of labour and 100 units of capital. It can sell the good at a constant net price (P), ie P is obtained after subtracting costs. Obviously we have simplified the problem because in this 'P' become profit per unit. Let P = 1.

Let x_1 and x_2 be the quantities of the goods produced by the processes 1 and 2 respectively. To maximize the profit P x_1 + P x_2, we write the objective function.

$\Pi = x_1 + x_2$ (since P = 1). The problem becomes

$$\textbf{Max } \pi = x_1 + x_2$$

Subject to: The labour constraint $\mathbf{8x_1 + x_2 \leq 200}$

The capital constraint $\mathbf{x_1 + x_2 \leq 100}$

And the non- negativity conditions $\mathbf{x_1 \geq 0, x_2 \geq 0}$

This is a problem in linear programming.

Example: Two foods F_1, F_2 are available at the prices of Rs. 1 and Rs. 2 per unit respectively. N_1, N_2, N_3 are essential for an individual. The table gives these minimum requirements and nutrients available from one unit of each of F_1, F_2. The question is of minimizing cost (C), while satisfying these requirements.

Nutrients	**Minimum requirements**	**One units of F_1**	**One units of F_2**
N_1	17	9	2
N_2	19	3	4
N_3	15	2	5

Total Cost (TC) $c = P_1 x_1 + P_2 x_2$ (x_1, x_2 quantities of F_1, F_2)

Where $P_1 = 1$, P = 2

We therefore have to Minimize

$C = x_1 + 2x_2$

Subject to the minimum nutrient requirement constraints,

$$9x_1 + x_2 \geq 17$$

$$3x_1 + 4x_2 \geq 19$$

$$2x_1 + 5x_2 \geq 15$$

Non- negativity conditions

$x_1 \geq 0$, $x_2 \geq 0$.

4.2.6 Graphical Solution

If the LPP consist of only two decision variable. We can apply the graphical method of solving the problem. It consists of seven steps, they are

1. Formulate the problem in to LPP.

2. Each inequality in the constraint may be treated as equality.

3. Draw the straight line corresponding to equation obtained steps (2) so there will be as many straight lines, as there are equations.

4. Identify the feasible region. This is the region which satisfies all the constraints in the problem.

5. The feasible region is a many sided figures. The corner point of the figure is to be located and they are coordinate to be measures.

6. Calculate the value of the objective function at each corner point.

7. The solution is given by the coordinate of the corner point which optimizes the objective function.

Example: Solve the following LPP graphically.

Maximize $Z = 3x_1 + 4x_2$

Subject to the constraints

$4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$

$x_1, x_2 \geq 0$

Treating the constraints are equal, we get

$4x_1 + 2x_2 = 80$ ....(1)

$2x_1 + 5x_2 = 180$… ....(2)

$x_1 = 0$… .... (3)

$x_2 = 0$ ....(4)

In equation (1), putting $x_1 = 0$, we get

$0x_1 + 2x_2 = 80$

$x_2 = 80/2$

$= 40$

When $x_2 = 0$

$4x_1 + 0x_2 = 80$

$x_1 = 80/4$

$= 20$

So (0, 40) and (20, 0) are the two point in the straight line given by equation (1)

Similarly in the equation (2), we get

$x_1 = 0$

$$0x_1 + 5x_2 = 180$$

$x_2 = 180/5$

$$= 36$$

$x_2 = 0$

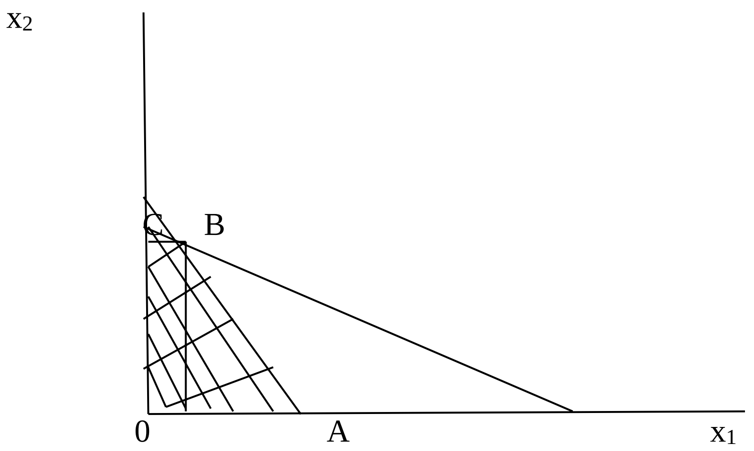
$$2x_1 + 0x_2 = 180$$

$x_1 = 180/2$

$$= 90$$

Therefore (0, 36) and (90, 0) are the two points on the straight line represent by the equation (2).

The equation (3) and (4) are representing the x_1 and x_2 axis respectively.



The feasible region is the (shaded area) shaded portion, it has four corner points, say OABC

The coordinate of O = (0, 0)

A = (20, 0)

C = (0, 36) and B can be obtained by solving the equations for the lines passing through that point.

The equations are (1) and (2)

$$4x_1 + 2x_2 = 80 \text{……… (1)}$$

$$2x_1 + 5x_2 = 180 \text{…...... (2)}$$

Then (2) – (1) $\quad 0 + 4x_2 = 140$

$x_2 = 140/4 = 35$

Substituting the value of x_2 in (1), we get

$x_1 = 40 – 35 / 2$

$$= 5/2$$

$= 2.5$

So the coordinate b are ($x_1 = 2.5$, $x_2 = 35$)

Evaluate the objective function of the corner points is given below.

Point	x_1 x_2	Z	
0	(0, 0)	0	
A	(20, 0)	60	$3 \times 20 + 4 \times 0 = 60$
B	(2.5, 35)	147.5	$3 \times 2.5 + 4 \times 35 = 147.5$
C	(0, 36)	144	$3 \times 0 + 4 \times 36 = 144$

B gives the maximum value of Z, so the solution is $x_1 = 2.5$ and $x_2 = 35$

Maximum value if Z = 147.5

Example: Solve the following LPP graphically.

Minimize $C = 6x_1 + 11x_2$

Subject to the constraints

$2x_1 + x_2 \geq 104$

$x_1 + 2x_2 \geq 76$

$x_1, x_2 \geq 0$

Treating the constraints are equal, we get

$2x_1 + x_2 = 104$ ....(1)

$x_1 + 2x_2 = 76$… .... (2)

$x_1 = 0$… ....(3)

$x_2 = 0$ ....(4)

In equation (1), putting x1 = 0, we get

$0x_1 + x_2 = 104$

$x_2 = 104$

When $x_2 = 0$

$2x_1 + 0x_2 = 104$

$2x_1 = 104/2$

$= 52$

So (0, 104) and (52, 0) are the two point in the straight line given by equation (1)

Similarly in the equation (2), we get

$x_1 = 0$

$$0x_1 + 2x_2 = 76$$

$$2\, x_2 = 76$$

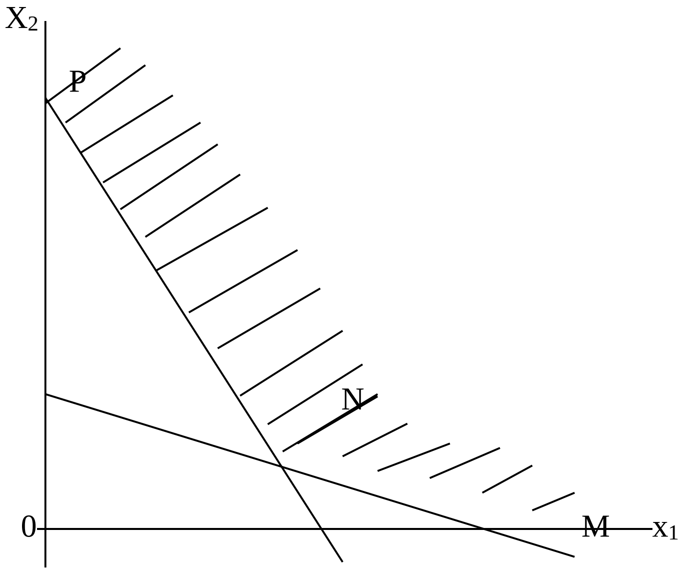
$x_2 = 76/2$

$$= 38$$

$x_1 + 0x_2 = 76$

$x_1 = 76$

Therefore (0, 38) and (76, 0) are the two points on the straight line represent by the equation (2).The equation (3) and (4) are representing the x_1 and x_2 axis respectively.



The feasible region is the (shaded area) shaded portion, it has three corner points, say PNM

The coordinate of M = (52, 0)

P = (0, 38)

N can be obtained by solving the equations for the lines passing through that point.

The equations are (1) and (2)

$$2x_1 + x_2 \geq 104 \ldots\ldots\ldots (1)$$

$x_1 + 2x_2 \geq 76 \ldots\ldots\ldots\ldots (2)$

Taking the second constraint and multiplied by 2 throughout the equation

$$2x_1 + x_2 = 104 \ldots\ldots\ldots\ldots (1)$$

$$2x_1 + 4x_2 = 152 \ldots\ldots\ldots (2)$$

Then (2) – (1) $\quad 0 + 3x_2 = 48$

$x_2 = 48/3$

$$= 16$$

Substituting the value of x2 in (1), we get

$2x_1 + 16 = 104$

$= 104 – 16/2$

$= 88/2$

$= 44$

So the coordinate b are (x_1 = 88, x_2 = 16)

Evaluate the objective function of the corner points is given below.

Point	x_1 x_2	Z	
P	(0, 104)	1144	$6\times 0 + 11\times 104=1144$
N	(46, 17)	440	$6\times 44+ 14 \times 16 = 440$
M	(75, 0)	450	$6 \times 75 + 11\times 0 = 450$

N gives the minimum value of C, so the solution is x_1 = 46 and x_2 = 17

Minimum value of C = 440

Exercise: A baker has 150 kilograms of flour, 22 kilos of sugar, and 27.5 kilos of butter with which to make two types of cake. Suppose that making one dozen A cakes requires 3 kilos of flour, kilo of butter, whereas making one dozen B cakes requires 6 kilos of flour, 0.5 kilo of sugar, and 1 kilo of butter. Suppose that the profit from one dozen A cakes is 20 and from one dozen B cakes is 30. How many dozen a cakes (x_1) and how many dozen B cakes (x_2) will maximize the baker's profit?

Solution:

An output of x_1 dozen and x_2 dozen B cakes would need $3x_1 + 6x_2$ kilos of flour. Because there are only 150 kilos of flour, the inequality.

$3x_1 + 6x_2 \leq 150$ (flour constraint).............(1)

Similarly, for sugar,

$x_1 + 0.5x_2 \leq 22$ (sugar constraint).............(2)

and for butter,

$x_1 + x_2 \leq 27.5$ (butter constraint)............(3)

Of course, $x_1 \geq 0$ and $x_2 \geq 0$. The profit obtained from producing x_1 dozen A cakes and x_2 dozen B cakes is

$Z = 20x_1 + 30x_2$ ....(4)

In short, the problem is to

$\text{Max } Z = 20x_1 + 30x_2$

s.t $3x_1 + 6x_2 \leq 150$

$x_1 + 0.5x_2 \leq 22$

$x_1 + x_2 \leq 27.5$

$x_1 \geq 0$

$x_2 \geq 0$

Exercise: Solve Graphically,

Max Z $3x_1 + 4x_2$

Sub to $3x_1 + 2x_2 \leq 6$

$x_1 + 4x_2 \leq 4$

$x_1 \geq 0,\ x_2 \geq 0$

Min C $10x_1 + 27x_2$

Sub to $x_1 + 3x_2 \geq 11$

$2x_1 + 5x_2 \geq 20$

$x_1 \geq 0,\ x_2 \geq 0$

Max Z $2x + 7y$

Sub to $4x + 5y \leq 20$

$3x + 7y \leq 21$

$x_1 \geq 0,\ x_2 \geq 0$

4.2.7 : Applications of linear programming

1. Production Management:

LP is applied for determining the optimal allocation of such resources as materials, machines, manpower, etc. by a firm. It is used to determine the optimal product- mix of the firm to maximize its revenue. It is also used for product smoothing and assembly line balancing.

2. Personnel Management:

LP technique enables the personnel manager to solve problems relating to recruitment, selection, training, and deployment of manpower to different departments of the firm. It is also used to determine the minimum number of employees required in various shifts to meet production schedule within a time schedule.

3. Inventory Management:

A firm is faced with the problem of inventory management of raw materials and finished products. The objective function in inventory management is to minimise inventory cost and the constraints are space and demand for the product. LP technique is used to solve this problem.

4. Marketing Management:

LP technique enables the marketing manager in analysing the audience coverage of advertising based on the available media, given the advertising budget as the constraint. It also helps the sales executive of a firm in finding the shortest route for his tour. With its use, the marketing manager determines the optimal distribution schedule for transporting the product from different warehouses to various market locations in such a manner that the total transport cost is the minimum.

5. Financial Management:

The financial manager of a firm, mutual fund, insurance company, bank, etc. uses the LP technique for the selection of investment portfolio of shares, bonds, etc. so as to maximise return on investment.

6. Blending Problem:

LP technique is also applicable to blending problem when a final product is produced by mixing a variety of raw materials. The blending problems arise in animal feed, diet problems, petroleum products, chemical products, etc. In all such cases, with raw materials and other inputs as constraints, the objective function is to minimise the cost of final product.

4.2.8 : Limitations of Linear Programming Technique

Linear programming has turned out to be a highly useful tool of analysis for the business executive. It is being increasingly made use of in theory of the firm, in managerial economics, in interregional trade, in general equilibrium analysis, in welfare economics and in development planning. But it has its limitations.

1. it is not easy to define a specific objective function.
2. Even if a specific objective function is laid down, it may not be so easy to find out various technological, financial and other constraints which may be operative in pursuing the given objective.
3. Given a specific objective and a set of constraints, it is possible that the constraints may not be directly expressible as linear inequalities.
4. Even if the above problems are surmounted, a major problem is one of estimating relevant values of the various constant coefficients that enter into a linear programming model, i.e., prices, etc.
5. This technique is based on the assumption of linear relations between inputs and outputs. This means that inputs and outputs can be added, multiplied and divided. But the relations between inputs and outputs are not always linear. In real life, most of the relations are non-linear.
6. This technique assumes perfect competition in product and factor markets. But perfect competition is not a reality.
7. The LP technique is based on the assumption of constant returns. In reality, there are either diminishing or increasing returns which a firm experiences in production.

8. It is a highly mathematical and complicated technique. The solution of a problem with linear programming requires the maximization or minimization of a clearly specified variable. The solution of a linear programming problem is also arrived at with such complicated method as the 'simplex method' which involves a large number of mathematical calculations.

It requires a special computational technique, an electric computer or desk calculator. Mostly, linear programming models present trial- and-error solutions and it is difficult to find out really optimal solutions to the various economic problems.

4.3 INPUT- OUTPUT MODELS

In a modern economy the production of one good requires the input of many other goods as intermediate goods in the production process. Leontief Input-output model is a technique to explain the general equilibrium of the economy. It is also known as "inter-industry analysis". Before analysing the input-output method, let us understand the meaning of the terms, "input" and "output". According to Professor J.R. Hicks, an input is "something which is bought for the enterprise" while an output is "something which is sold by it." An input is obtained but an output is produced. Thus input represents the expenditure of the firm, and output its receipts. The sum of the money values of inputs is the total cost of a firm and the sum of the money values of the output is its total revenue.

The input-output analysis tells us that there are industrial interrelationships and inter-dependencies in the economic system as a whole. The inputs of one industry are the outputs of another industry and vice versa, so that ultimately their mutual relationships lead to equilibrium between supply and demand in the economy as a whole.

Leontief input-output analysis answers what level of output each of *n* industries in an economy should produce that will just be sufficient to satisfy the total demand for the product. The total demand x for product *i* will be the summation of all intermediate demand for the product plus the final demand b for the product arising from consumers, investors, the government and exporters as ultimate users. For example, the output of steel industry is needed as an input in many other industries, or even for that industry itself; therefore the correct level of steel output will depend on the input requirements of all the *n* industries.in turn the output of many other industries wil enter into the steel industry as inputs, and consequently the "correct" levels of other products will in turn depend partly on the input requirements of the steel industry. In view of inter industry dependence, any set of "correct" output levels for *n* industries must be one that is consistent withal the input requirements in the economy, so that no bottlenecks will raise anywhere. If a_{ij} is a technical coefficient expressing the value of input i required to produce one dollar's worth of product *j* the total demand for product *i* can be expressed as

$x_i = a_{i1}x_1+a_{i2}x_2+\ldots\ldots+ a_{in}x_n+ bi$

fori= 1,2,….,n. in matrix form

$$X = AX+B$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

A is the matrix of technical coefficients. To find the level of total output needed to satisfy final demand, we can for X in terms of matrix of technical coefficients and the column vector of final demand.

$$X - AX = B$$

$$(I-A)\, X = B$$

$$X = (I-A)^{-1} B$$

I-A is called the Leontieff matrix.

4.3.1 Assumptions

This analysis is based on the following assumptions:

(i) The whole economy is divided into two sectors—“inter-industry sectors” and “final-demand sectors,” both being capable of sub-sectoral division.

(ii) The total output of any inter-industry sector is generally capable of being used as inputs by other inter-industry sectors, by itself and by final demand sectors.

(iii) No two products are produced jointly. Each industry produces only one homogeneous product.

(iv) Prices, consumer demands and factor supplies are given.

(v) There are constant returns to scale.

(vi) There are no external economies and diseconomies of production.

(vii) The combinations of inputs are employed in rigidly fixed proportions. The inputs remain in constant proportion to the level of output. It implies that there is no substitution between different materials and no technological progress. There are fixed input coefficients of production.

Example 1. Determine the total demand x for industries 1,2,3, given the matrix of technical coefficients A and the final demand vector B.

$$A = \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0.5 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.1 \end{bmatrix} \qquad B = \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix}$$

$$X = (I-A)^{-1} B$$

$$I-A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0.5 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.4 & -0.1 \\ -0.5 & 0.8 & -0.6 \\ -0.1 & -0.3 & 0.9 \end{bmatrix}$$

$(I-A)^{-1} = \frac{1}{IAI}$ Adj A

$$= \frac{1}{0.151}\begin{bmatrix} 0.54 & 0.3 & 0.32 \\ 0.51 & 0.62 & 0.17 \\ 0.23 & 0.25 & 0.36 \end{bmatrix}$$

$$X = \frac{1}{0.151}\begin{bmatrix} 0.54 & 0.3 & 0.32 \\ 0.51 & 0.62 & 0.47 \\ 0.23 & 0.25 & 0.36 \end{bmatrix}\begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix} = \frac{1}{0.151}\begin{bmatrix} 24.3 \\ 30.5 \\ 17.9 \end{bmatrix} = \begin{bmatrix} 160.3 \\ 201.99 \\ 118.54 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 2: $A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$, $B = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.2 \\ -0.1 & -0.3 & 0.8 \end{bmatrix}$$

$$(I-A)^{-1} = \frac{1}{IAI}\text{ Adj } A = \frac{1}{0.384}\begin{bmatrix} 0.66 & 0.30 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.60 \end{bmatrix}$$

$$X = \frac{1}{0.384}\begin{bmatrix} 0.66 & 0.30 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.60 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 24.84 \\ 20.68 \\ 18.36 \end{bmatrix}$$

4.3.2 Limitations of Input Output Analysis

Major limitations faced by input-output analysis are as follows:

1. Its framework rests on Leontiefs basic assumption of constancy of input co-efficient of production which was split up above as constant returns of scale and technique of production. The assumption of constant returns to scale holds good in a stationary economy, while that of constant technique of production in stationary technology.

2. Assumption of fixed co-efficient of production ignores the possibility of factor substitution. There is always the possibility of some substitutions even in a short period, while substitution possibilities are likely to be relatively greater over a longer period.
3. The assumption of linear equations, which relates outputs of one industry to inputs of others, appears to be unrealistic. Since factors are mostly indivisible, increases in outputs do not always require proportionate increases in inputs.
4. The rigidity of the input-output model cannot reflect such phenomena as bottlenecks, increasing costs, etc.
5. The input-output model is severely simplified and restricted as it lays exclusive emphasis on the production side for the economy. It does not tell us why the inputs and outputs are of a particular pattern in the economy.
6. Another difficulty arises in the case of “final demand” or “bill of goods”. In this model, the purchases by the government and consumers are taken as given and treated as a specific bill of goods. Final demand is regarded as an independent variable. It might,

therefore, fail to utilize all the factors proportionately or need more than their available supply. Assuming constancy of co-efficiency of production, the analysis is not in a position to solve this difficulty.

7. There is no mechanism for price adjustments in the input-output analysis which makes it unrealistic. "The analysis of cost-price relations proceeds on the assumption that each industrial sector adjusts the price of its output by just enough to cover the change in the case of its primary and intermediate output."
8. The dynamic input-output analysis involves certain conceptual difficulties. First, the use of capital in production necessarily leads to stream of output at different points of time being jointly produced. But the input-output analysis rules out joint production. Second, it cannot be taken for investment and output will necessarily be non-negative.
9. The input-output model thrives on equations that cannot be easily arrived at. The first thing is to ascertain the pattern of equations, then to find out the necessary voluminous data. Equations presuppose the knowledge of higher mathematics and correct data are not so easy to ascertain. This makes the input-output model abstract and difficult.

MODULE V

MARKET EQUILIBRIUM

Market Equilibrium: Perfect Competition -Monopoly - Discriminating Monopoly

5.1. Market Equilibrium

Equilibrium literally means balance. It means a position from which there is no tendency to change. The forces which determine it can said to be in balance at the equilibrium point. Unless these forces change, the equilibrium would not change. Equilibrium price, then, is the price at which the forces which determine the price are in balance. The two forces which determine the price of any commodity in the market are demand and supply. Geometrically, this is the price where the demand and the supply curves cross. If we let D (p) be the market demand curve and S (p) the market supply curve, the equilibrium price is the price p* that solves the equation.

D (p) = S (p)

The solution to this equation, p*, is the price where market demand equals market supply. When market is in equilibrium, then there is no excess demand and supply.

Assuming that both demand and supply curves are linear, demand – supply model can be stated in the form of the following equation.

QD = a-bp…............(1)

QS = c+dp…..............(2)

QD = QS.............(3)

Where QS and QS are quantities demanded and supplied respectively, a and c are intercept coefficients of demand and supply curves respectively, b and d are the coefficients that measures the slop of these curves, equation (3) is the equilibrium condition.

Thus in equilibrium

a - bp = c + dp

a – c = dp + bp = p (d+b)

Dividing both sides by d+b we have

$\frac{a-c}{d+b}$= p $\frac{d+b}{d+b}$

Or equilibrium price

P =$\frac{a-c}{d+b}$ …….........................(4)

Substituting (4) into (1) we have equilibrium quantity

QD = a-b $\frac{a-c}{a+b}$

=$\frac{ad+bc}{b+d}$……………………….(5)

Equation (4) and (5) describe the qualitative results of the model. If the values of the parameters a, b, c and d are given we can obtain the equilibrium price and quantity by substituting the values of these parameters in the qualitative results of equation (4) and (5).

Example: Suppose the following demand and supply functions of a commodity are given which is being produced under perfect competition. Find out the equilibrium price and quantity.

QD = 10 – 4p… .... (1)

QS = -2 + 8p… .... (2)

Solution: There are two alternatives ways of solving for equilibrium price and quantity.

First we can find out the equilibrium price and quantity by using the equilibrium condition, namely

QD =QS .... (3)

Second, we can obtain equilibrium price and quantity by using the qualitative results of the demand and supply model.

$P = \frac{a-c}{d+b}$ and $QD = \frac{ad+bc}{b+d}$

1. Since in equilibrium QD=QS

$10 - 4p = -2 + 8p$

$-2 + 8p + 4p - 10 = 0$

$12 = 12p$

$P = \frac{12}{12} = 1$

P=1..................................(4)

Substituting the equilibrium price of p* =1 either into demand equation (1) or supply equation (2), we get the equilibrium quantity QD of

$QD = 10 - 4(1) = 6$

$QS = -2 + 8(1) = 6$

Alternative Method:

$P = \frac{a-c}{d+b}$

Where a= 10, b= 4, c= -2, d =8

$P = \frac{10-(-2)}{8+4} = \frac{12}{12} = 1$

$QD = \frac{ad+bc}{b+d},$

$= \frac{10*8 + 4*(-2)}{4+8}$

$= \frac{80 + (-8)}{12}$

$= \frac{72}{12} = 6$

5.2. Equilibrium in the Perfect Competitive Market.

Perfect competition is a market situation where the market is automatically regulated by the forces of demand and supply over which individual sellers have no control.In the perfect competitive market the firms are in equilibrium when they maximize their profits (Π). The profit is the difference between the total cost and total revenue, i.e. Π = TR-TC

The conditions for equilibrium are

1. MC = MR

2. Slop of MC > slope of MR

Derivation of the equilibrium of the firm

The firms aims at the maximization of its profit

Π = TR-TC

Where

Π =Profit

R =Total Revenue

C =Total cost

Clearly R = $f_1(X)$ and C = $f_2(X)$, given the price P.

(a) The first-order condition for the maximization of a function is that its first derivative (with respect to X in our case) be equal to zero. Differentiating the total-profit function and equating to zero we obtain

$$\frac{\delta \Pi}{\delta X} = \frac{\delta R}{\delta X} - \frac{\delta C}{\delta X} = 0$$

or $\frac{\delta R}{\delta X} - \frac{\delta C}{\delta X}$

The term $\frac{\delta R}{\delta X}$ is the slope of the total revenue curve, that is, the marginal revenue. The term $\frac{\delta C}{\delta X}$ is the slope of the total cost curve, or the marginal cost. Thus the firs-order condition for profit maximization is

MR = MC

Given that MR > 0, MR must also be positive at equilibrium. Since MR = P the first-order condition may be written as MC = P.

(b) The second-order condition for a maximum requires that the second derivative of the function be negative (implying that after its highest point the curve turns downwards). The second derivative of the total-profit function is

$$\frac{d^2\Pi}{dX^2} = \frac{d^2R}{dX^2} - \frac{d^2C}{dX^2}$$

This must be negative if the function has been maximized, that is

$\frac{d^2R}{dX^2} - \frac{d^2C}{dX^2} < 0$

which yields the condition?

$$\frac{d^2R}{dX^2} < \frac{d^2C}{dX^2}$$

But $\frac{d^2R}{dX^2}$ is thope of MR curve and $\frac{d^2C}{dX^2}$ is thope of the MC curve. Hence the second-order condition may verbally be written as follows

(slope of MR) < (slope of MC)

Thus the MC must have a steeper slope than the MR curve or the MC must cut the MR curve from below. In pure competition the slope of the MR curve is zero, hence the second-order condition is simplified as follows.

$$0 < \frac{d^2C}{dX^2}$$

Which reads: the MC curve must have a positive slope, or the MC must be rising.

Example: A perfectly competitive market faces P = Rs. 4 and TC = $X^3 - 7X^2 + 12X + 5$.

Find the best level of output of the firm. Also find the profit of the firm at this level of output.

First condition requires, MR = MC

TR = PX = 4X, as P = 4

MR = $\frac{dTR}{dX}$ = 4, which is also equal to price. So MR = 4 = P

MC = $\frac{dTC}{dX} = 3X^2 - 14X + 12$

Setting MR = MC and solving for X to find the critical values

$4 = 3X^2 - 14X + 12$

$3X^2 - 14X + 12 - 4 = 0$

$3X^2 - 14X + 8 = 0$

By factorization we have the values as

(3X – 2) and (X – 4)

So 3X = 2, X = $\frac{2}{3}$ and X = 4

This means that at the equilibrium point MR = MC, X = $\frac{2}{3}$ and X = 4

The second condition requires that MC must be rising at this point of intersection. In other words, the slope of the MC curve should be positive at the point where MC = MR. the equation for the slope of the MC curve is to find its derivatives.

$\frac{dMC}{dX} = 6X - 14$

Then substitute the two critical values X =-and X = 4 in the above equation to find out the point

which maximize the profit.

When X =$\frac{2}{3}$, 6X – 14 would be 6$\frac{2}{3}$ - 4 = -10. It is not the profit maximizing output.

When X = 4, 6X – 14 would be 6×4 – 14 = 10.

Here the profit is maximized when the output is equal to 10 units.

Then we have to find the maximum profit. The maximum profit is obtained when the output is at 10 units. So substitute the value, i.e, X=4 in the profit function.

Then Π = TR– TC

$\Pi = 4X - (X^3 - 7X^2 + 12X + 5)$

$= 4X - X^3 - 7X^2 - 12X - 5$

$= -X^3 - 7X^2 - 8X - 5$

$= -64 + 112 - 32 - 5$

$= 11$

The firm maximizes its profit at the output level of 4 units and at this level its maximum profit is Rs.11.

5.3. Equilibrium in the Monopoly

Monopoly is the form of market organization in which a single firm sells a commodity for which there are no close substitutes. Thus, the monopolist represents the industry and faces the industry’s negatively sloped demand curve for the commodity. As opposed to a perfectly competitive firm, a monopolist can earn profits in the long run because entry into the industry is blocked or very difficult. And the monopolist has complete control over price.

A. Short-run Equilibrium

The monopolist maximizes his short-run profit if the following two conditions are fulfilled:

1. The MC is equal to the MR. i,e, MC = MR

2. The slope of the MC is greater than the slope of the MR at the point of intersection.

Mathematical derivation

The given demand function is X = g(P)

Which may be solved for P, P = $f_1(X)$

The given cost function is C = $f_2(X)$

The monopolist aims at the maximization of his profit

$$\Pi = TR - TC$$

(a) The first- order condition for maximum profit Π

$\frac{\delta \Pi}{\delta X} = 0$

$\frac{\delta \Pi}{\delta X} = \frac{\delta R}{\delta X} - \frac{\delta C}{\delta X} = 0$

Or

$\frac{\delta R}{\delta X} = \frac{\delta C}{\delta X}$

That is MR = MC

(b) The second- order condition for maximum profit

$\frac{d^2\Pi}{dX^2} < 0$

$$\frac{d^2\Pi}{dX^2} = \frac{d^2R}{dX^2} - \frac{d^2C}{dX^2} < 0$$

$$or \ \frac{d^2R}{dX^2} < \frac{d^2C}{dX^2}$$

that is

$\begin{bmatrix} slope \\ of\ MR \end{bmatrix} < \begin{bmatrix} slope \\ of\ MC \end{bmatrix}$

Example

Given the demand curve of the monopolist

X = 80 – 0.2P

Which may be solved for P

P =100 -8X

Given the cost function of the monopolist

C = 50 + 20X

The goal of the monopolist is to maximum profit

Π = R – C

i. We first find the MR

R = XP = X (100-8X)

R = 100X - 8X^2

MR = $\frac{\delta R}{\delta X}$ = 100 – 16X

ii. We next find the MC

C = 50 + 20X

MC = $\frac{\delta C}{\delta X}$ = 20

iii. We equate MR and MC

MR = MC

100 – 16X = 20

X = 5

iv. The monopolist's price is found by substituting x = 5 into the demand- price equation

$$P = 100 – 8X$$
$$= 100 – 8*5$$
$$= 60$$

v. The profit is

$$\Pi = R – C = 100X - 8X^2 – 50 + 20X$$
$$= 100*5 - 8*5^2 - 50 + 20*5$$
$$= 500 - 200 - 50 + 100$$
$$= 150$$

This profit is the maximum possible, since the second- order condition is satisfied :

(a) From $\frac{\delta C}{\delta X} = 20$

$$\frac{\delta^2 C}{\delta X^2} = 0$$

(b) From $\frac{\delta R}{\delta X} = 100 – 16X$

We have $\frac{\delta^2 R}{\delta X^2} = -16$

Clearly $-16 < 0$.

Alternative Method

The same problem can be worked out by another method. After finding TR and TC, compute the profit function Π.

$$\Pi = TR – TC = 100X - 8X^2 – (50 + 20X)$$
$$= 100X - 8X^2 – 50 + 20X$$
$$= 80X - 8X^2 – 50$$
$$\Pi = -8X^2 + 80X – 50$$

As per the optimization rule, we can optimize the function.

At first find the first order derivative and equate it with zero and find the critical value.

$$\Pi' = - 16X + 80X = 0$$
$$= -16X + 80 = 0,$$
$$= -16X = - 80$$
$$= -16 * 5 = -80$$

$$X = 5$$

The second order condition for the maximization must be less than zero.

$$\Pi'' = -16 < 0$$
$$\Pi'' (X=5) = -16 < 0$$

Here the conditions for the maximization have been satisfied. So the function is maximized at X = 5

To find the price, P, substitute X = 15 in the demand function P = 100 – 8X

i.e, P = 100 – 8(5) = 100 – 40 = 60

The monopolist maximize his profit when the quantity X = 5 and the price, P = 60.

The maximum profit of the monopolist,

$$\Pi = -8X^2 + 80X - 50$$

Substitute $\Pi = -8(5^2) + 30(5) - 50$

$= -200 + 350 = 150$

Example:

The monopolist's demand curve is

$$X = 200 - 2p, \text{ or } p = 100 - 0.5X$$

The costs of the two plants are

$C_1 = 10X_1$ and $= 0.25X_2 2$

The goal of the monopolist is to maximize profit

$$\Pi = R - C_1 - C_2$$

1. $R = XP = X(100 - 0.5X)$

   $R = 100X - 0.5X^2$

   $MR = \frac{\delta R}{\delta X} = 100 - X = 100 - (X_1 + X_2)$

2. $C_1 = 10X_1$

   $MC_1 = \frac{\delta C_1}{\delta X_1} = 10$

and

$C_2 = 0.25X_2 2$

$MC_2 = \frac{\delta C_2}{\delta X_2} = 0.5X_2$

3. Equating each MC to the common MR

   $100 - X_1 - X_2 = 10$

   $100 - X_1 - X_1 = 0.5X_2$

   Solving for X_1 and X_2 we find

$X_1 = 70$ and $X_2 = 20$

So that the total X is 90 units. This total output will be sold at price P defined by

$P = 100 - 0.5X = 55$

The monopolist's profit is

$\Pi = R - C_1 - C_2$

$= 4950 - 10(20) - 0.25(4900)$

$\Pi = 3525$

This is the maximum profit since the second-order condition is fulfilled.

5.4. Discriminating Monopoly

Price discrimination means the practice of selling the same commodity at different prices to different buyers. Price discrimination is not possible in ordinary competitive conditions unless there is some product differentiation. But a monopolist can charge different prices from different customers for the same commodity associated with monopoly. It is a technique by which the monopolist makes the consumers pay according to their ability. Mrs. Joan Robinson defines price discrimination as "the act of selling the same article produced under a single control at different prices to different consumers'.

Mathematical derivation of the equilibrium position of the price-discriminating monopolist

Given the total demand of the monopolist

$$P = f(X)$$

Assume that the demand curves of the segmented markets are

$P_1 = f_1(X_1)$ and $P_2 = f_2(X_2)$

The cost of the firm is

$$C = f(X) = f(X_1 + X_2)$$

The firm aims at the maximization of its profit

$$\Pi = R_1 + R_2 - C$$

The first- order condition for profit maximization requires

$\frac{\delta \Pi}{\delta X_1} = 0$ and $\frac{\delta \Pi}{\delta X_2} = 0$

(a) $\frac{\delta \Pi}{\delta X_1} = \frac{\delta R_1}{\delta X_1} - \frac{\delta C}{\delta X_1} = 0$ and $\frac{\delta \Pi}{\delta X_2} = \frac{\delta R_2}{\delta X_2} - \frac{\delta C}{\delta X_2} = 0$

(b) $\frac{\delta R_1}{\delta X_1} - \frac{\delta C}{\delta X_1}$ or $MR_1 = MC_1$; and $\frac{\delta R_2}{\delta X_2} = \frac{\delta C}{\delta X_2}$ or $MR_2 = MC_2$

But

$$MC_1 = MC_2 = MC = \frac{dC}{dX}$$

Therefore

$$MC = MR_1 = MR_2$$

The second-order condition for profit maximization requires

$\frac{\delta^2 R_1}{\delta X_1} < \frac{d^2 C}{dX^2}$ and $\frac{\delta^2 R^2}{\delta X_2 2} < \frac{d^2 C}{dX^2}$

That is, the MR in each market must be increasing less rapidly than the MC for the output as a whole.

Example

Assume that the total demand is X = 50 – 0.5P (or P = 100 - 2x).

Assume further that the demand functions of segmented markets are

X_1 = 32- 0.4P_1 or P_1= 80 – 2.5X_1

X_2 = 18- 0.1P_2 or P_2= 180 – 10X_2

(Clearly $X_1 + X_2 = X$)

Finally, assume that the cost function is

C = 50 + 40X = 50 + 40 ($X_1 + X_2$)

The firm aims at the maximization of its profit

$\Pi = R_1 + R_2 - C$

Solution

(1) $R_1 = X_1 P_1 = X_1 (80 - 2.5X_1) = 80X_1 - 2.5\, X_1 2$

$MR_1 = \frac{\delta R_1}{\delta X_1} = 80 - 5X_1$

(2) $R_2 = X_2 P_2 = X_2 (180 - 10X_2) = 1\,80X_2 - 10\, X_2 2$

$MR_2 = \frac{\delta R_2}{\delta X_2} = 180 - 20X_2$

(3) MC = $\frac{\delta C}{\delta X_1} = \frac{\delta C}{\delta X_2} = \frac{\delta C}{\delta X} = 40$

Setting the MR in each market equal to the common MC we obtain

$$80 - 5X_1 = 40$$
$$180 - 20X_2 = 40$$
$$X_1 = 8$$
$$X_1 = 7$$

X= 15

The prices are

P_1 = 80 – 2.5X_1 = 60

P_2 = 180 – 10X_2 = 110

The profit is

$\Pi = R_1 + R_2 - C = 500$

The elasticities are

$e_1 = \frac{\delta X_1}{\delta P_1} \times \frac{P_1}{X_1} = (0.4)\frac{60}{8} = 3$

$e_2 = \frac{\delta X_2}{\delta P_2} \times \frac{P_2}{X_2} = (0.1)\frac{110}{7} = 1.57$

Thus $e_1 < e_2$ and $P_1 < P_2$.

Comparing the above results with those for the example of the simple monopolist we observe that X is the same in both cases but the Π of the discriminating monopolist is larger.

Price discrimination and the price elasticity of demand

$$MR = P\left(1 - \frac{1}{e}\right)$$

In the case of price discrimination we have

$$MR_1 = P_1\left(1 - \frac{1}{e_1}\right)$$

$$MR_2 = P_2\left(1 - \frac{1}{e_2}\right)$$

And $MR_1 = MR_2$

Therefore

$$P_1\left(1 - \frac{1}{e_1}\right) = P_2\left(1 - \frac{1}{e_2}\right)$$

Or

$$\frac{P_1}{P_2} = \frac{\left(1 - \frac{1}{e_2}\right)}{\left(1 - \frac{1}{e_1}\right)}$$

Where e_1 = elasticity of D_1

e_2 = elasticity of D_2

If $e_1 = e_2$ the ratio of prices is equal to unity :

$$\frac{P_1}{P_2} = 1$$

That is, $P_1 = P_2$. This mean that when elasticities are the same price discrimination is not profitable. The monopolist will charge a uniform price for his product. If price elasticities differ price will be higher in the market whose demand is less elastic. This is obvious from the equality of the MR's

$$P_1\left(1 - \frac{1}{e_1}\right) = P_2\left(1 - \frac{1}{e_2}\right)$$

If $|e_1| > |e_1|$, then

$$\left(1 - \frac{1}{e_1}\right) > \left(1 - \frac{1}{e_2}\right)$$

Thus for the equality of MR's to be fulfilled

$P_1 < P_2$

That is, the market with the higher elasticity will have the lower price.

MATHEMATICAL ECONOMICS SYLLABUS

a. Introduction

Mathematical economics is an approach to economic analysis where mathematical symbols and theorems are used. Modern economics is analytical and mathematical in structure. Thus the language of mathematics has deeply influenced the whole body of the science of economics. Every student of economics must possess a good proficiency in the fundamental methods of mathematical economics. One of the significant developments in Economics is the increased application of quantitative methods and econometrics. A reasonable understanding of econometric principles isindispensable for further studies in economics.

b. Objectives

Course is aimed at introducing students to the most fundamental aspects of mathematical economics and econometrics. The objective is to develop skills in these. It also aims at developing critical thinking, and problem-solving, empirical research and model building capabilities.

c. Learning Outcome

The students will acquire mathematical skills which will help them to build and test models ineconomics and related fields. The course will also assist them in higher studies in economics.

d. Syllabus

MATHEMATICAL ECONOMICS

Module I. Introduction to Mathematical Economics *(10 % weightage)*

Mathematical Economics: Meaning and Importance-Mathematical Representation of Economic Models- Economic functions: Demand function, Supply function, Utility function, Consumption function, Production function, Cost function, Revenue function, Profit function, Saving function, Investment function

Module II. Marginal Concepts *(25 % weightage)*

Marginal utility, Marginal propensity to Consume, Marginal propensity to Save, Marginal product, Marginal Cost, Marginal Revenue, Marginal Rate of Substitution, Marginal Rate of Technical Substitution. Relationship between Average Revenue and Marginal Revenue-Relationship between Average Cost and Marginal Cost -Elasticity: Price elasticity, Income elasticity, Cross elasticity.

Module III. Optimisation *(25 % weightage)*

Optimisation of single / multi variable functions -Constrained optimisation with Lagrange Multiplier –Significance of Lagrange Multiplier. Economic applications: Utility Maximisation, Cost Minimisation, Profit Maximisation.

Module IV Production Function, Linear Programming and Input Output analysis *(25 % weightage)*

Production function- homogeneous and non-homogeneous. Degree of homogeneity and returns to scale - Properties of Cobb-Douglas production function. Production possibility curve. Linear programming: –Basic concept, Nature of feasible, basic and optimal solution; Graphic solution –The Dual -Applications of linear programming in economics. Input-output analysis – Matrix of technical coefficients –The Leontief matrix –computation of total demand for a two/ three sector economy.

Module V. Market Equilibrium *(15 % weightage)*

Market Equilibrium: Perfect Competition - Monopoly Discriminating Monopoly

Reference:

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3. Henderson, J. M. and R.E. Quandt (1980), Microeconomic Theory: A Mathematical Approach, McGraw Hill, New Delhi.
4. James Bradfield , Jeffrey Baldani, An Introduction to Mathematical Economics, Cengage Learning India Pvt Ltd (2008)
5. A. Koutsoyiannis, Modern Microeconomics, Palgrave Macmillan; 2nd Revised edition edition (2003)*(– see mathematical appendices for each topic given at the bottom of the page)*